CHAPTER VIII

ATMOSPHERIC STATICS

In Chapter VII we have studied the behavior of individual air parcels when they undergo certain physical transformations. We now turn our attention to several general aspects related to the vertical stratification of the atmosphere and then, in the next chapter, to the vertical stability of the atmosphere. By 'vertical' we mean that the atmosphere will be considered above a certain location on the Earth's surface, generally taking into account neither the horizontal motions due to the Earth's rotation and to horizontal gradients of pressure, nor the large-scale vertical motions.

In this chapter we shall study hydrostatic equilibrium, several cases of ideal atmospheres, and the calculation of heights. We shall use only one coordinate z, in the vertical direction, increasing upwards, with its origin at mean sea level. We shall assume that the state variables remain constant for a constant z; that is, the isobaric, isothermal and equal humidity surfaces will be horizontal.

8.1. The Geopotential Field

Every system in the atmosphere is subject to the force of gravity. This is the resultant of two forces: (1) the gravitational attraction per unit mass $\mathbf{f}$, in accordance with Newton's universal law of gravitation, and (2) the centrifugal force, which results from choosing our frame of reference fixed to the rotating Earth; this much smaller component is equal (per unit mass) to $\omega^2\mathbf{r}$, where ω is the angular velocity and r the distance from the axis of rotation. The vector sum $\mathbf{g}$ is the force of gravity per unit mass, or simply *gravity* (see Figure VIII-1):

$$\mathbf{g} = \mathbf{f} + \omega^2\mathbf{r}. \tag{1}$$

If we call R the radius of the Earth (i.e., the distance from the center of mass to the surface), we must take into account that R varies with the latitude φ, due to the ellipticity of the Earth. The values are maximum and minimum at the Equator and at the Poles, respectively:

$$R_{\text{Equator}} = 6378.1 \text{ km}$$

$$R_{\text{Pole}} = 6356.9 \text{ km}.$$

From the inverse-square law of universal gravitation, it follows that the variation of $f = |\mathbf{f}|$ with altitude over a given location on the Earth's surface can be expressed by

$$f_z = f_0 \frac{R^2}{(R + z)^2} \tag{2}$$

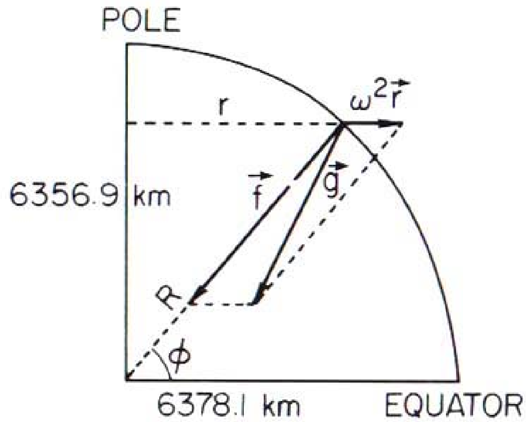

Fig. VIII-1. Forces in the geopotential field.

where f_0 is the value of f at mean sea level and f_z is the value at an altitude z above mean sea level. Thus

$$f_z = f_0\left(1 + \frac{z}{R}\right)^{-2} = f_0\left(1 - 2\frac{z}{R} + \dots\right) \cong f_0\left(1 - 2\frac{z}{R}\right). \tag{3}$$

For the purpose of introducing R into Equation (3), an average value can be used and thus

$$f_z \cong f_0(1 - 3.14 \times 10^{-7}\, z) \tag{4}$$

with sufficient approximation (z in meters).

The gravity g at sea level varies with latitude due to the centrifugal force and to the ellipticity. As $\omega^2 r \ll f$, the angle between **g** and **f** is very small and we can write from the cosine theorem (see Figure VIII-1)

$$g_{\varphi,0} = f_0\left[1 + \left(\frac{\omega^2 r}{f_0}\right)^2 - 2\frac{\omega^2 r}{f_0}\cos\varphi\right]^{\frac{1}{2}} \cong f_0 - \omega^2 r\cos\varphi \tag{5}$$

where $g_{\varphi,0}$ means gravity at latitude φ and sea level. Or, considering that $r = R\cos\varphi$

$$g_{\varphi,0} = f_0 - \omega^2 R\cos^2\varphi\,. \tag{6}$$

At $\varphi = 45°$,

$$g_{45,0} = f_0 - \tfrac{1}{2}\omega^2 R\,. \tag{7}$$

Eliminating f_0 between Equations (6) and (7), we find

$$g_{\varphi,0} = g_{45,0} - \tfrac{1}{2}\omega^2 R\cos 2\varphi\,. \tag{8}$$

As R is approximately constant, Equation (8) indicates that the centrifugal term is proportional to $\cos 2\varphi$; it amounts to -1.7 cm s^{-2} at the equator, 0 at 45° and $+1.7$ cm s^{-2} at the poles.

The variation of g with latitude due to ellipticity includes the effect of variation of distance from the mass center (which attains a maximum of 21 km between the equator and the poles) and the effect of a non-spherical distribution of mass. The former would imply a difference in gravity of 6.6 cm s^{-2} between the poles and equator, and the latter partially compensates this difference. Both may be taken into account together with the centrifugal term by adjusting the coefficient of $\cos 2\varphi$ in Equation (8). Thus the following approximate formula for the dependence of g on latitude may be written:

$$g_{\varphi,0} = g_{45,0}(1 - 0.00259 \cos 2\varphi). \tag{9}$$

We consider now that because g differs little from f the same correction factor for altitude of formula (4) (expression between brackets) can be applied to it:

$$g_{\varphi,z} = g_{\varphi,0}(1 - 3.14 \times 10^{-7} z) \tag{10}$$

and introducing Equation (9), we finally obtain

$$g_{\varphi,z} = g_{45,0}(1 - a_1 \cos 2\varphi)(1 - a_2 z) \tag{11}$$

where

$$a_1 = 2.59 \times 10^{-3}$$

$$a_2 = 3.14 \times 10^{-7}\ \mathrm{m}^{-1}$$

$$g_{45,0} = 9.80616\ \mathrm{m\ s}^{-2}.$$

As Equation (11) indicates, gravity at mean sea level varies from 9.78 m s^{-2} at the equator to 9.83 m s^{-2} at the poles.

More complicated expressions have been developed and must be used for accurate estimates of the gravity, which also depends slightly on local topography. Meteorologists use the so-called *meteorological gravity system*, which gives values very slightly different from the *Potsdam system*, widely used in geodesy. These differences need not concern us here. More accurate formulas than (11) to calculate local values of the acceleration of gravity can be consulted in the WMO Tables (see Bibliography).

A *standard value* at sea level g_0 has also been adopted for reference. It is

$$g_0 = 9.80665\ \mathrm{m\ s}^{-2}. \tag{12}$$

If a unit mass moves in the gravitational field, the force of gravity performs a work

$$\delta w = \mathbf{g} \cdot \mathrm{d}\mathbf{r} = g \cos\theta\ \mathrm{d}r \tag{13}$$

where d**r** is the displacement, θ the angle it forms with **g**, and the dot between the vectors indicates a scalar product.

Experience shows that when the mass returns to its original position

$$\oint \mathbf{g} \cdot \mathrm{d}\mathbf{r} = 0. \tag{14}$$

This property is expressed by saying that the gravitational field is *conservative*. It follows from Equation (14) that δw is an exact differential that defines w as a point function. It is preferred, however, to define it with opposite sign, and it is called the *geopotential* ϕ. The conservatism of the gravitational field is also expressed by saying that $\mathbf{g}$ is a force derived from a potential.

We shall have, in general

$$\begin{aligned} \phi &= \phi(x, y, z), \\ \mathrm{d}\phi &= -\,\mathbf{g}\cdot\mathrm{d}\mathbf{r} \text{ (exact differential)}, \\ \Delta\phi &= -\int \mathbf{g}\cdot\mathrm{d}\mathbf{r}, \end{aligned} \tag{15}$$

where $\Delta\phi$ is independent of the integration path. As ϕ depends only on the altitude z,* these expressions simplify to

$$\begin{aligned} \phi &= \phi(z) \\ \mathrm{d}\phi &= g\,\mathrm{d}z \\ \Delta\phi &= \int g\,\mathrm{d}z \cong g\Delta z\,. \end{aligned} \tag{16}$$

The second one says that the gravity is given by the geopotential gradient. The last approximate equation indicates that, as g varies little with z, it may be taken for many purposes as a constant.

As a point function, ϕ is defined except for an additive constant. We fix it by choosing zero geopotential at mean sea level:

$$\phi(0) = 0\,, \tag{17}$$

so that

$$\phi(z) = \int_0^z g\,\mathrm{d}z \cong gz\,. \tag{18}$$

8.2. The Hydrostatic Equation

In a state of equilibrium, the force of gravity is everywhere balanced by the pressure forces, whose resultant is opposite to it. Let us consider a layer of thickness $\mathrm{d}z$ in a column of unit area (Figure VIII-2). A force p acts on its base, directed upwards,

* If z is measured along a line of gravity force, i.e., along a line parallel at every point to $\mathbf{g}$, it will follow a slightly curved line. This deviation from a straight line is however negligible for most purposes.

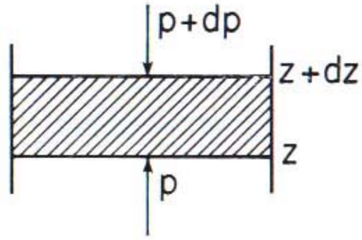

Fig. VIII-2. Pressure variation in vertical.

while a downward force $p+\mathrm{d}p$ acts on the top. Therefore a net force $-\mathrm{d}p$ is acting upwards on a mass $\varrho\,\mathrm{d}z$. The force of gravity is $g\varrho\,\mathrm{d}z$. As they must cancel:

$$\mathrm{d}p = -\,g\varrho\,\mathrm{d}z$$

or

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\,g\varrho\,. \tag{19}$$

This is the hydrostatic equation for the simple case when $p=p(z)$. The isobaric surfaces, as well as the equipotential surfaces, are considered here to be horizontal.

If we compare Equations (16) and (19), we have for hydrostatic equilibrium the equivalent expressions:

$$\mathrm{d}p = -\,\varrho\,\mathrm{d}\phi \tag{20}$$

and

$$\mathrm{d}\phi = -\,v\,\mathrm{d}p\,. \tag{21}$$

Integration of Equation (21) gives:

$$\Delta\phi = -\int_1^2 v\,\mathrm{d}p = -\int_1^2 RT\,\mathrm{d}\ln p = -\,R_\mathrm{d}\int_1^2 T_\mathrm{v}\,\mathrm{d}\ln p\,. \tag{22}$$

In order to perform this integration, we must know the virtual temperature T_v as a function of the pressure p. We shall consider this problem in Section 5 for a particular kind of atmosphere, and in Section 11 for the general case.

8.3. Equipotential and Isobaric Surfaces. Dynamic and Geopotential Height

The geopotential field may be described by the equipotential surfaces; the thickness, h_ϕ, of a layer of unit geopotential difference, $\Delta\phi$, will be given by:

$$\Delta\phi = gh_\phi = 1 \text{ unit of geopotential.} \tag{23}$$

ϕ has dimensions of energy per unit mass or specific energy. In the MKS system the units are J $kg^{-1} = m^2 s^{-2}$. Thus, with $g \cong 9.8$ m s^{-2}, Equation (23) gives

$$h_\phi \cong 0.102 \text{ m} . \tag{24}$$

Therefore approximately 1.02 m (the exact value depending on that of g) correspond to a difference in geopotential of 10 J kg^{-1}; this equivalence suggested at one time the use of another unit for the specific energy, called the *dynamic metre*, which may be abbreviated dyn-m and was defined by

$$1 \text{ dyn-m} = 10 \text{ J kg}^{-1} . \tag{25}$$

The dynamic kilometre (dyn-km) or other similar units could also be used; they bear the same relations among themselves as the corresponding units of length. The value of ϕ in these units has in the past been frequently referred to as the *dynamic height* or *dynamic altitude*. They are seldom encountered in meteorology at the present time.

It is now customary in meteorology to express the geopotential as *geopotential altitude*. As a physical quantity this is again identically the geopotential, but expressed in units called *standard geopotential metres* or, more simply, *geopotential metres* (gpm). The conversion factor has the numerical value of the standard gravity $|g_0| = 9.80665$:*

$$1 \text{ gpm} = |g_0| \text{ J kg}^{-1} \tag{26}$$

Thus the geopotential can be written

$$\phi = \int_0^z g \, \mathrm{d}z \, (\text{J kg}^{-1}) = \frac{1}{|g_0|} \int_0^z g \, \mathrm{d}z \, (\text{gpm}) = \frac{\bar{g}}{|g_0|} z \, (\text{gpm}) \tag{27}$$

and is called 'geopotential altitude' if expressed in gpm or any multiple of this unit (such as gpkm = geopotential kilometre). In the last expression, $\bar{g}$ is the average value of g in the integration interval. As $\bar{g}/|g_0|$ is numerically very close to unity (within a fraction of 1% anywhere in the troposphere), it is obvious from Equation (27) that the numerical values of the geopotential in gpm (i.e., the geopotential altitude) and of the altitude in m are almost equal.

Summarizing the equivalent expressions of geopotential in different units, we have:

$$\phi = \bar{g}z \text{ J kg}^{-1} = \frac{\bar{g}}{|g_0|} z \text{ gpm} = \frac{\bar{g}}{10} z \text{ dyn-}m \tag{28}$$

where $\bar{g}z = \int_0^z g \, \mathrm{d}z$ and z is given in metres.

It must be stressed that neither the dynamic meter nor the geopotential metre are units

* Until 1971, the convention used in meteorology was to take a conversion factor equal to 9.8 (J kg^{-1}) gpm^{-1} (exactly).

of length, but units of geopotential, i.e., of specific energy. Neither is the 'dynamic altitude' or the 'geopotential altitude' a length, but identically the geopotential, the word 'altitude' merely referring to the type of specific energy units used (dyn-m or gpm, respectively).

If we introduce the expression (10) to calculate the average value g between mean sea level and z, we obtain

$$\bar{g} = \frac{1}{z}\int_0^z g\,\mathrm{d}z = \frac{g_{\varphi,0}}{z}\int_0^z (1 - 3.14 \times 10^{-7} z)\,\mathrm{d}z$$

$$= g_{\varphi,0}(1 - 1.57 \times 10^{-7} z) \tag{29}$$

and

$$\phi = \frac{g_{\varphi,0}}{|g_0|} z(1 - 1.57 \times 10^{-7} z) \tag{30}$$

(z in metres).

In general, the ratio $\bar{g}/|g_0|$ in formula (27) will differ from unity by less than 0.2%. It is customary in meteorology to use the geopotential height rather than geometrical height for the representation of the state of the free atmosphere, for a number of reasons, theoretical as well as practical. In the first place, in most dynamic equations the differential of height is associated with the acceleration due to gravity as a product, and $g\,\mathrm{d}z$ can always be replaced by $|g_0|\,\mathrm{d}\phi$ with ϕ in geopotential altitude units. This is particularly true if isobaric coordinates (x, y, p, t) are used (as is now standard), where position in the vertical is defined by the pressure. In the second place, true or geometrical height is never required and is seldom measured; this is partly because of the central position of pressure (or related parameters) as an indicator of position in the vertical. Finally, the computational advantages of ϕ over z virtually necessitate the use of geopotential height.

When the foot is used as a unit of length, the exact equivalences 1 ft = 0.3048 m and 1 gpft = 0.3048 gpm are adopted, where gpft stands for *geopotential foot*.

The thickness h_p, between isobaric surfaces separated by a unit pressure difference, can be similarly calculated from Equation (19):

$$\Delta p = g\varrho h_p = 1 \text{ unit of pressure}$$

$$h_p = \frac{1}{g\varrho} = \frac{v}{g}. \tag{31}$$

But in this case v, and therefore h_p, vary rapidly with p, and therefore with height. At 1000 mb, v is about 0.8 $\mathrm{m^3\,kg^{-1}}$, and $h_p = 0.08$ m (for 1 Pa = 0.01 mb); it follows that in the layers near to the ground, the pressure drops with height at a rate of 100 mb for 800 m. As we rise in the atmosphere, the unit layers become thicker.

By comparing Equations (23) and (31), we obtain

$$h_p = \mathrm{v}h_\phi. \tag{32}$$

8.4. Thermal Gradients

The vertical temperature gradient or lapse rate may be defined by the derivative

$$\beta = -\,\mathrm{d}T/\mathrm{d}z\,. \tag{33}$$

However, it is also convenient to define it in the atmosphere with respect to the geopotential:

$$\gamma = -\frac{\mathrm{d}T}{\mathrm{d}\phi} = -\frac{1}{g}\frac{\mathrm{d}T}{\mathrm{d}z} = \frac{\beta}{g}. \tag{34}$$

We shall use this last definition.

It should be noticed that altough both parameters are proportional and have close numerical values (if dynamic or geopotential heights are used for ϕ), they are two physical quantities of different kind. The first definition (β) gives the decrease in the absolute temperature with height, and its units are K m^{-1} (in MKS system). γ gives the decrease of temperature with the geopotential, and is measured in K s^2 m^{-2} in MKS system, although it is customarily expressed, for convenience, in degrees per unit of geopotential height (K gpm^{-1} or K gpft^{-1}).

Starting from the virtual temperature, a lapse rate of virtual temperature γ_v is similarly defined:

$$\gamma_v = -\frac{\mathrm{d}T_v}{\mathrm{d}\phi}. \tag{35}$$

Other expressions for γ_v may be derived if we take into account Equations (18), (21) and the gas law:

$$\gamma_v = -\frac{1}{g}\frac{\mathrm{d}T_v}{\mathrm{d}z} = \frac{1}{v}\frac{\mathrm{d}T_v}{\mathrm{d}p} = \frac{1}{R_d}\frac{\mathrm{d}\ln T_v}{\mathrm{d}\ln p}. \tag{36}$$

We remark now that these derivatives and differentials may refer: (1) to the variations undergone by an air mass during a process, for instance during an adiabatic ascent, and (2) to the variations in the values of the static variables along the vertical for an atmosphere at rest. We shall call the first ones *process derivatives* or *process differentials*, while we shall refer to the latter as *geometric derivatives* or *geometric differentials*. We shall only be concerned in this chapter with geometric variations.

8.5. Constant-Lapse-Rate Atmospheres

We shall now consider the case of an atmosphere with constant lapse rate γ_v (of virtual temperature), for which T_v decreases proportionally with ϕ. This is a particularly important case inasmuch as it is the obviously simplest way to approximate a real atmosphere. As in general, in the troposphere, temperature decreases with height, γ_v is usually a positive quantity.

We write for $z=0$ (mean sea level):

$$T_v = T_{v_0}, \qquad p = p_0, \qquad \phi = 0.$$

We find $T_v = f(\phi)$ by integrating Equation (35) from mean sea level to any height:

$$T_v = T_{v_0} - \gamma_v \phi$$

or

$$\frac{T_v}{T_{v_0}} = 1 - \frac{\gamma_v \phi}{T_{v_0}} \tag{37}$$

i.e., the virtual temperature decreases linearly with the geopotential, or, if we neglect the small variation of g, with the altitude.

From Equation (36) we obtain:

$$\mathrm{d} \ln T_v = R_d \gamma_v \, \mathrm{d} \ln p, \tag{38}$$

which contains the hydrostatic equation, and integrated gives $T_v = f(p)$:

$$\frac{T_v}{T_{v_0}} = \left(\frac{p}{p_0}\right)^{R_d \gamma_v} \tag{39}$$

and by eliminating (T_v/T_{v_0}) between Equations (37) and (38), we obtain $p = f(\phi)$:

$$p = p_0 \left[1 - \frac{\gamma_v \phi}{T_{v_0}}\right]^{1/R_d \gamma_v}. \tag{40}$$

Thus, Equations (37), (39) and (40) relate the variables T_v, p and ϕ. Equations (39) and (40) are the result of integrating the hydrostatic equation for the particular case $\gamma_v = \text{const}$. It may be noticed that, dimensionally, $[\gamma_v] = 1/[R_d]$.

If we consider dry air (for which $T_v = T$) and compare formula (39) with Chapter II, formula (61), we see that the formulas are equivalent if we set $k = (n-1)/n = R_d \gamma_v$, i.e., the formulas for an atmosphere with constant lapse rate are similar to those for polytropic processes. However, they describe here the geometric distribution of temperature and pressure rather than variations during a process.

We can see from Equations (37) and (40) that p and T_v become 0 for $\phi_1 = T_{v_0}/\gamma_v$. This is therefore called the *limiting geopotential height of an atmosphere with constant lapse rate* (we prefer this terminology to the simpler term geopotential height, which is open to some ambiguity). For this idealized model atmosphere, there will not be any air for $\phi > \phi_1$.

We shall now consider three special cases of constant-lapse-rate atmospheres.

8.6. Atmosphere of Homogeneous Density

If $\gamma_v = 1/R_d = 34.2$ K gpkm^{-1}, from Equation (38) and the gas law relation d ln $T_v =$

$= \mathrm{d}\ln p + \mathrm{d}\ln v$, it follows that $\mathrm{d}v = 0$. This atmosphere has therefore a homogeneous density. It is sometimes called the 'homogeneous atmosphere', although only density is constant, while temperature and pressure vary with height.

Equations (37), (39) and (40) simplify to

$$T_v = T_{v_0} - \phi/R_d \tag{41}$$

$$T_v/T_{v_0} = p/p_0 \tag{42}$$

$$p = p_0(1 - \phi/R_d T_{v_0}) \tag{43}$$

where the second one is simply the gas law for constant density; and the limiting geopotential height is

$$\phi_1 = R_d T_{v_0}.$$

If $T_{v_0} = 273\,\mathrm{K}$, $\phi_1 = 7990$ gpm. We shall see later that the lapse rate never reaches this value in the real atmosphere, except for very shallow layers over a strongly heated surface.

8.7. Dry-Adiabatic Atmosphere

We consider now an atmosphere with a lapse rate:

$$\gamma_v = 1/c_{p_d} = \gamma_d = 9.76\ \mathrm{K\ gpkm^{-1}}.$$

Equation (39) becomes

$$T_v = T_{v_0}\left(\frac{p}{p_0}\right)^{\varkappa_d} \tag{44}$$

which coincides with one of Poisson's equations describing the adiabatic expansion of dry air. The geometric distribution of temperature for this static atmosphere is therefore described by the curve followed by the adiabatic expansion process of a parcel of dry air. It will be characterized by the virtual potential temperature:

$$\theta_v = T_{v_0}\left(\frac{1000\ \mathrm{mb}}{p_0}\right)^{\varkappa_d} \tag{45}$$

which is constant throughout the whole atmosphere, as can be verified by introducing Equation (44) into the expression of θ_v for any T_v, p.

Equations (37) and (40) become:

$$T_v = T_{v_0} - \phi/c_{p_d}, \tag{46}$$

and

$$p = p_0\left(1 - \frac{\phi}{c_{p_d} T_{v_0}}\right)^{1/\varkappa_d} \tag{47}$$

and the limiting geopotential height is:

$$\phi_l = c_{p_d} T_{v_0},$$

If $T_{v_0} = 273\,\mathrm{K}$, $\phi_l = 27\,950$ gpm.

8.8. Isothermal Atmosphere

We assume:

$$\gamma_v = 0, \qquad T_v = T_{v_0} = \text{const.}$$

Integration of Equation (21) gives:

$$\phi = -\int v\,\mathrm{d}p = -R_d T_{v_0} \ln \frac{p}{p_0} \tag{48}$$

or

$$p = p_0\,\mathrm{e}^{-\phi/R_d T_v} = p_0\,\mathrm{e}^{-\phi/RT} \tag{49}$$

Therefore, in this case, pressure decreases exponentially with height, and tends to 0 when $z \to \infty$. The limiting geopotential height of this atmosphere is thus infinite.

The expression (49) can also be written

$$p = p_0\,\mathrm{e}^{-gM_d z/R^* T_v} = p_0\,\mathrm{e}^{-gMz/R^* T} = p_0\,\mathrm{e}^{-z/H} \tag{50}$$

where M_d is the average molecular weight of dry air, M is the average molecular weight of the air in the atmosphere and we have defined the parameter

$$H = \frac{R^* T_v}{g M_d} = \frac{R^* T}{g M} \tag{51}$$

which is called the *scale height* and represents the altitude at which the ground pressure becomes reduced by the factor e^{-1}. H is a constant for an isothermal atmosphere, if the variation of g is neglected. When T varies with height, H can no longer be considered constant. In that case one can speak of a 'local' scale height; for instance, the geopotential height of the atmosphere of homogeneous density (Section 6) corresponds to the value at the ground of the scale height: $H_l = \phi_l/g = R^* T_{v_0}/gM$.

8.9. Standard Atmosphere

Of the particular cases which we have considered, the atmosphere of homogeneous density has only a theoretical interest, and the isothermal atmosphere is only applicable to layers with a constant temperature. In the lower stratosphere, this is frequently a useful approximation, especially at high latitudes, but in the troposphere is seldom valid except for relatively thin layers. In this latter domain, the adiabatic atmosphere is of more practical importance, since it gives, as we shall see, an upper limit for the value of the lapse rate of a vertically stable atmosphere, and has also the temperature distribution of a vertically-mixed layer (cf. Chapter VII, Section 12). But the real atmosphere has always on the average lower values of γ and, although this is in general not constant with height, we may define an atmosphere with a constant lapse rate which approximates the real average case (for the troposphere).

Thus several 'standard atmospheres' have been defined, which are particularly important in aeronautics, where they are used for reference as an approximation to the real atmosphere, and for calibrating and using altimeters.

We give here the basic definitions of the standard atmosphere adopted by the International Civil Aviation Organization (ICAO). These are stated using an altitude H' corrected for the variation of g, according to the expression

$$H = \frac{1}{g_0}\phi = \frac{1}{g_0}\int_0^z g\,dz. \tag{52}$$

Here g_0 is the standard value of gravity, while g is the actual value, depending on latitude and altitude. The formula is similar to Equation (27), but here g_0 retains its dimensions as an acceleration, and therefore H is a real length. Obviously H is numerically equal to the geopotential ϕ expressed in geopotential height units. As we have seen in Section 1, g decreases by about 0.3% for a 10 km increase of altitude. If we disregard the variations of g, $H \cong z$. The conditions defining the standard atmosphere are:

(1) Atmosphere of pure dry air with constant chemical composition in the vertical, with mean molecular weight $M = 28.9644$ (C^{12} scale).

(2) Ideal gas behavior.

(3) Standard sea-level value of the acceleration due to gravity: $g_0 = 9.806\,65\ \text{m s}^{-2}$.

(4) Hydrostatic equilibrium.

(5) At mean sea level, the temperature is $T_0 = 15\,^\circ\text{C} = 288.15\,\text{K}$ and the pressure $p_0 = 1013.25\ \text{mb} = 1\ \text{atm}$.

TABLE VIII-1
Standard atmosphere

p(mb)	H(m)	T(°C)
1013.25	0	15.0
1000	110	14.3
900	990	8.6
800	1 950	2.3
700	3 010	− 4.6
600	4 200	− 12.3
500	5 570	− 21.2
400	7 180	− 31.7
300	9 160	− 44.5
226.3	11 000	− 56.5

(6) For values of H up to 11 000 m above mean sea level (tropopause) the lapse rate is constant and given by $\beta_0 \doteq -\mathrm{d}T/\mathrm{d}H = 6.5\ \mathrm{K\ km^{-1}}$.

(7) For altitudes $H \geqslant 11\,000$ m (stratosphere) and up to 20 000 m, the temperature is constant and equal to $-56.5\,^\circ\mathrm{C}$. Then the lapse rate becomes $-1.0\ \mathrm{K\ km^{-1}}$, up to 32 000 m.

Table VIII-1 gives some values of p, H and T for this standard atmosphere.

According to Equations (37), (39) and (40), the relations between T, p and ϕ (or H), are given for the standard atmosphere up to 11 000 m by

$$T = T_0 - \beta_0 H = 288.15 - 6.5 \times 10^3 H \tag{53}$$

$$T = T_0\left(\frac{p}{p_0}\right)^{R_a\beta_0/g_0} = 288.15\left(\frac{p}{1013.25}\right)^{0.1903} \tag{54}$$

$$p = p_0\left(1 - \frac{\beta_0 H}{T_0}\right)^{g_0/R_a\beta_0}$$

$$= 1013.25(1 - 2.255 \times 10^{-5} H)^{5.256} \tag{55}$$

where T is given in K, p in mb and H in m. It may be noticed that, as H and the geopotential altitude are numerically equal, and so are γ_0 and β_0, if the former is the corresponding lapse rate of the standard atmosphere expressed in $\mathrm{K\ gpm^{-1}}$, ϕ in gpm can be written for H in the previous formulas, without further change ($\beta_0 H = \gamma_0 \phi$, where $\gamma_0 = 6.5$ K $\mathrm{gpkm^{-1}}$).

8.10. Altimeter

Because of its practical importance, we shall describe the altimeter used in airplanes to determine the height at which they are flying, and the corrections that must be made to its readings.

The altimeter is an aneroid barometer with two scales, which we shall call the *main scale* and the *auxiliary scale*. They correspond basically to two different ways of measuring the pressure. A mechanical transmission measures the effect of pressure on the barometer by the position of hands moving over the main scale. This is schematically represented in Figures VIII-3a–c by the position of a pointer. An alternative way of measuring the pressure is by using the auxiliary scale and a compensation method. The auxiliary scale is actually covered except for a window with an index showing the reading matched to the zero of the main scale (see figure). It is graduated in pressure units, in such a way that the reading is the correct pressure on the barometer when the hands read zero on the main scale. This situation is obtained by shifting the scales appropriately, and is indicated in Figure VIII-3a; in the real instrument the main scale and the window are actually fixed and it is the auxiliary scale and the hands that rotate simultaneously (rather than the main scale, as shown for convenience in the figure). The real scales of the instrument look like the sketch in Figure VIII-3d.

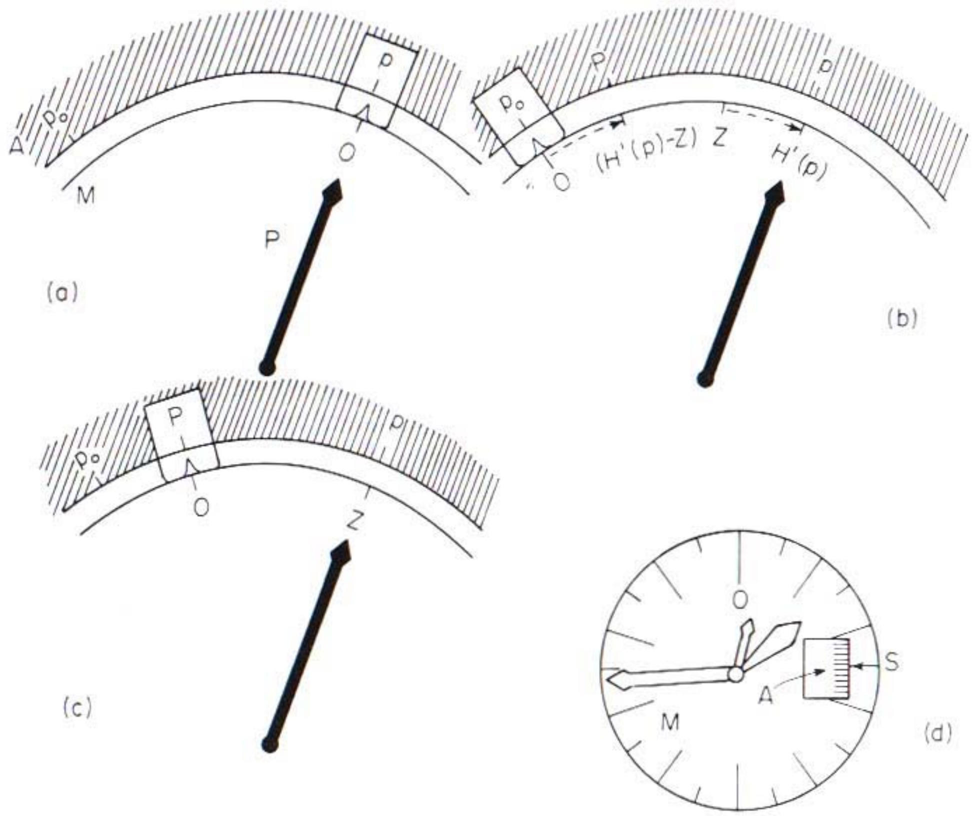

Fig. VIII-3. Altimeter scales.

The reading on the auxiliary scale matched to the zero on the main scale is called the *altimeter setting* or *Kollsman number*; this is p, p_0 and P in Figure VIII-3a, b and c, respectively.

The main scale is not graduated in pressure units, but in altitude units, related to the pressure by Equation (55). Therefore, if the altimeter setting is $p_0 = 1013.25$ mb, the hands will show the correct altitude on the main scale when used in a standard atmosphere. In particular, they will show zero altitude at 1013.25 mb. This is the setting in Figure VIII-3b. The reading $H(p)$, corresponding to any pressure p, is called the *pressure altitude* or *standard altitude*. It is the correct altitude of the isobar p in the standard atmosphere and coincides therefore with the value of H from Equation (55). For any particular setting different from p_0, the hands on the main scale at any moment indicate the difference of actual pressure from that of the setting, in altitude units H; this reading will be the real difference in altitude if the atmosphere is standard.

If the altimeter is set at p_0 while being used in an atmosphere different from the standard, in that zero altitude corresponds to another pressure P, the readings will be in error. Thus if the real altitude of a station with pressure p is Z, clearly $Z \neq H(p)$ unless the atmosphere is standard. In order to match the readings Z and p, we have to displace the zero altitude (Figure VIII-3c) so that it will correspond to a pressure P in the standard atmosphere for an altitude $(H(p) - Z)$. If an aircraft obtains information about the value of P from an airport and sets its altimeter accordingly, the reading will give the altitude of the airport when landing.

The procedure for obtaining the altimeter setting P is therefore:

(1) Obtain $H(p)$ from the actual pressure p at the station, by using a standard atmosphere table or Equation (55).

(2) Subtract the true altitude of the field Z.

(3) Obtain the altimeter setting P from $(H(p) - Z)$ through a standard atmosphere table or Equation (55).

This pressure correction is equivalent to approximating the atmosphere over the station by a standard atmosphere where a constant has been added to all pressures, the constant being chosen so as to make p correspond to Z.

Conversely, if instead of setting the altimeter to P, it is set with zero altitude at p_0, the correction to be added to any reading z above the station would be $D = z - H(p)$, sometimes called the 'D factor'.

The pressure correction has to be determined carefully as the values of z are very sensitive to small pressure variations. According to Equation (55), an error of 1 mb in p will give an error of 8 to 9 m in z.

To explain the temperature correction, let us assume that the station is at sea level, and its temperature is T_0' instead of $T_0 = 15\,°\mathrm{C}$. The average lapse rate of the atmosphere β up to the flying level will also be different in general from the standard value β_0. If the altimeter is set at the standard pressure p_0, its reading z' will be given by Equation (55), which may be written

$$p = p_0\left(1 - \frac{\beta_0 z'}{T_0}\right)^{g_0/R_d\beta_0}. \tag{56}$$

Here p is the actual pressure and the difference between actual (z) and corrected (H) altitudes is neglected, for simplicity. If we approximate the real atmosphere by an atmosphere with constant lapse rate β, the relation between the pressure p and the real altitude z will be given (from Equation (40)) by

$$p = p_0\left(1 - \frac{\beta z}{T_0'}\right)^{g_0/R_d\beta}.$$

Equating these two expressions and solving for z, we obtain

$$z = \frac{T_0'}{\beta}\left[1 - \left(1 - \frac{\beta_0 z'}{T_0}\right)^{\beta/\beta_0}\right]. \tag{57}$$

If z is not large, $\beta_0 z'/T_0$ and $\beta z/T_0'$ may be considered small relative to unity,* the exponential within the bracket in Equation (57) can be developed and only the first order term retained:

* Even for large z this ratio will not exceed about 0.25.

$$z = \frac{T'_0}{\beta}\left[1 - 1 + \frac{\beta}{\beta_0}\frac{\beta_0 z'}{T_0} - \ldots\right] \cong \frac{T'_0 z'}{T_0} \tag{58}$$

or

$$z = z'\left(1 + \frac{\Delta T}{T_0}\right) \tag{59}$$

which gives the correction by a simple formula ($\Delta T = T'_0 - T_0$).

It may be remarked that differences in lapse rate from the standard value have little influence on the correction, because they only enter in second-order terms in the previous development, so that they do not appear in the first-order approximation (59).

This correction may be in appreciable error if there are discontinuities in the atmosphere, as when the aircraft is flying above a frontal or a subsidence inversion. It should also be remarked that the temperature correction is much less critical than the pressure one, because it is proportional to the altitude and thus it becomes zero on landing, whereas an error in the setting will give an additive difference at all altitudes. With respect to the error during flight, for a given difference of pressure between ground and the aircraft, the altitude over the ground will be larger than the indication if $T'_0 > T_0$, because the intermediate atmosphere is less dense than the standard.

8.11. Integration of the Hydrostatic Equation

Formulas (39) and (40) give the result of integrating the hydrostatic equation for the particular case in which γ_v is a constant. The important problem arises now of finding a convenient method for obtaining the pressure (or the temperature) as a function of the geopotential ϕ (or of the altitude z) in any real atmosphere; that is, a convenient method for integrating the hydrostatic equation. This is easily done with an aerological diagram, as we shall see, by a summation over successive atmospheric layers.

Let us consider formula (22):

$$\Delta\phi = -\int_1^2 v\,\mathrm{d}p = -\int_1^2 RT\,\mathrm{d}\ln p = -R_\mathrm{d}\int_1^2 T_\mathrm{v}\,\mathrm{d}\ln p\,. \tag{60}$$

This is the same integral that appears in Chapter VI, Equation (17), whose graphical determination was explained in Chapter VI, Section 11. It should be realized that its meaning is now different. Here $\Delta\phi$ depends on the geometric structure, so that $\mathrm{d}p$ is a change of pressure with height in a static atmosphere; in Chapter VI we were considering a reversible process undergone by an air parcel. However, the value of the integral will be the same in both cases, provided the path of the parcel is represented by the same curve describing the geometric change of properties in the static atmosphere. Therefore, we can apply the same methods described before. Thus

$$-\int_1^2 T_\mathrm{v}\,\mathrm{d}\ln p = \textstyle\sum_\mathrm{em} \tag{61}$$

according to Chapter VI, formula (20) and with the same meaning of the area $\sum_{\rm em}$, which can be determined by any of those methods. The mean isotherm method will give

$$\Delta\phi = -R_{\rm d}\bar{T}_{\rm v}\ln\frac{p_2}{p_1} = R_{\rm d}\sum\nolimits_{\rm em} \tag{62}$$

(where we might substitute $\overline{RT}$ for $R_{\rm d}\bar{T}_{\rm v}$). For each pair of chosen isobars p_1, p_2 this depends only on $\bar{T}_{\rm v}$ and can be printed for convenience on the diagram. The mean adiabat method gives the formula (cf. Equations (60), (61) and Chapter VI, Equation (26)):

$$\begin{aligned}\Delta\phi &= c_{\rm p}(T_1''' - T_2''') \cong c_{\rm p_d}(T_1''' - T_2''')\\ &= 1005(T_1''' - T_2''')\ {\rm J\ kg^{-1}} = 102.5(T_1''' - T_2''')\ {\rm gpm}\end{aligned} \tag{63}$$

where T_1''' and T_2''', as in Chapter VI, Section 11, are the temperatures of the intersections of the mean adiabat with the isobars p_1 and p_2.

If a tephigram is used, we could apply again the relation

$$\begin{aligned}\Delta\phi &= -\int_1^2 v\,{\rm d}p = -\Delta h + q = -c_{\rm p}\Delta T + c_{\rm p}\int_1^2 T\,{\rm d}\ln\theta = \\ &= c_{\rm p}(-\Delta T + \sum\nolimits_{\rm te})\end{aligned} \tag{64}$$

(cf. Chapter VI, Equations (18) and (28)). But it will be more convenient to use the same methods as for the emagram, as explained in Chapter VI, Section 11 (cf. Chapter VI, Equation (31)).

With these procedures, the curve $\phi = {\rm f}(p)$ or $z = {\rm f}(p)$ can be computed and plotted on a diagram, on the basis of the state curve $T = {\rm f}(p)$ representing the vertical structure of the atmosphere (the sounding). The procedure will follow the construction of a table such as Table VIII-2. The starting value will be the known altitude $z_{\rm s}$ of the station, where the pressure is $p_{\rm s}$. The sounding is then divided in layers between succes-

TABLE VIII-2
Computation of altitude

i	p_i	$\Delta_i z$	$z_i = z_s + \sum_1^i \Delta_i z$
0	p_s		z_s
		$\Delta_1 z$	
1	p_1		z_1
		$\Delta_2 z$	
2	p_2		z_2
		$\Delta_3 z$	
3	p_3		z_3
		–	
–	–		–

sive isobars p_i, generally at intervals of 50 or 100 mb. Equation (60), applied to the first layer between p_s and p_1, will give $\Delta_1\phi$ and therefore $\Delta_1 z = z_1 - z_s$; this computation is performed graphically by any of the procedures described. The altitude of the isobar p_1 will be given by $z_1 = z_s + \Delta_1 z$. Then the second graphical integration is performed on the second layer; the result, $\Delta_2 z$, added to z_1, will give the altitude z_2 of the isobar p_2. And the procedure is continued along the whole sounding. The altitudes z_i can then be plotted on the same diagram, as a function of p_i, using an auxiliary scale for z.

The altitude curve has only a small curvature on an emagram or a tephigram, because the temperature varies slowly and thus ϕ or z is roughly proportional to $\ln p$. The vertical dimension, normal to isobars on the diagram, is therefore roughly proportional to height in the atmosphere.

It must be emphasized that these calculations are to be performed with the curve of virtual temperature as a function of pressure. In aerological practice, the relative humidity at any level is converted to the corresponding mixing ratio, and from the latter is computed the so-called virtual temperature increment, $\Delta T_v = T_v - T$, which is given from Equations (72) and (75) of Chapter IV, with sufficient accuracy, as $\Delta T_v \cong 0.6 rT$.

It may be of interest to compute what would be the pressure at mean sea level at the location of a station of height ϕ_s, should the atmosphere extend to $\phi = 0$; or to estimate the pressure at any other close level. This problem arises in routine meteorological practice in the production of mean-sea-level (m.s.l.) pressure values for a 'surface' chart. Since pressure values (at a constant level) are useful in the diagnosis of wind fields (at that level) and in the synoptic appreciation of weather systems, it is necessary to 'reduce' ground pressures to a reference level (chosen as mean sea level) in order to construct meaningful isobars. The procedure to be described below is reasonably satisfactory if station heights and temperatures are themselves relatively smooth fields in the horizontal, especially at elevations not in excess of 1 km. Let us consider the above problem, assuming that $\phi_s > 0$. The computation is made by considering a fictitious atmosphere from $\phi = 0$ to ϕ_s; to it is assigned a lapse rate equal to $\gamma_d/2$, as a reasonable mean value. The temperature T_0 and pressure p_0 at mean sea level are then computed from

$$T_0 = T_s + \frac{\gamma_d}{2}\phi_s \tag{65}$$

$$p_0 = p_s\left(1 + \frac{\gamma_d \phi_s}{2T_s}\right)^{2/\varkappa_d}. \tag{66}$$

Here it is customary to take for T_s an average value in such a way as to avoid the strong variations close to the ground, due to its daily warming and cooling by radiation. These variations are shown in Figure VIII-4. T_s is taken as the average between the actual value and that of 12 hours before:

$$T_s = \frac{T_{s,\,\text{actual}} + T_{s,\,-12\,\text{hs}}}{2}. \tag{67}$$

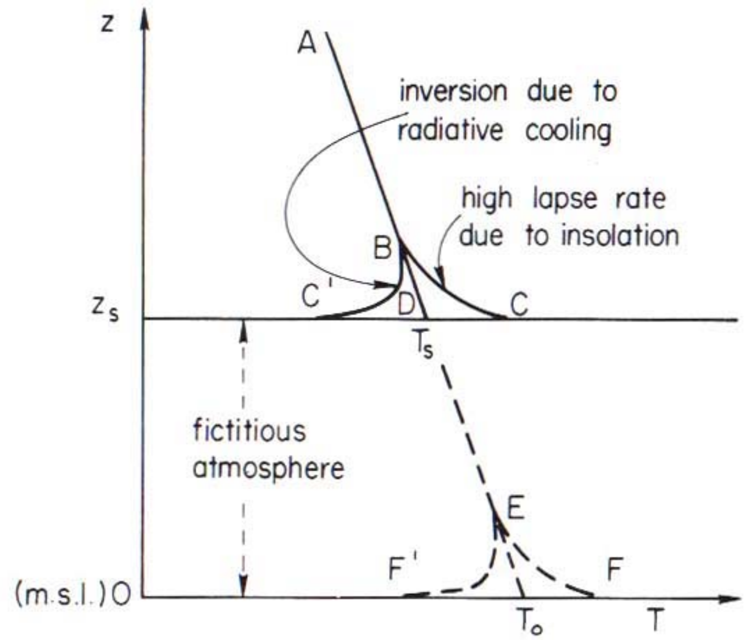

Fig. VIII-4. Computation of 'mean-sea-level pressures' by considering a fictitious atmosphere.

The additional temperature for reduction to m.s.l. is given by the dotted line of lapse rate $\gamma_d/2$. To compensate for the substitution of BC or BC′ by BD, which alters the contribution of this layer to the pressure, a profile ABDEF or ABDEF′ is assumed, whereby the peculiar stratification near the ground is transferred to the fictitious ground at sea level.

PROBLEMS

1. Find the dry adiabatic lapse rate, in K gpkm^{-1}, for an atmosphere composed entirely of Ar. The atomic weight of Ar is 39.9.
2. An aircraft is flying at 1000 m of altitude. The temperature T_v is 15 °C and the pressure p, 894 mb. (a) Assuming that the mean lapse rate γ_v below the aircraft is 8.5 K gpkm^{-1}, find the pressure p_0 at mean sea level. (b) Derive an expression to estimate the relative error in p_0 for a given error in γ_v. (c) What is the error in p_0 if the real mean lapse rate if 6.5 K gpkm^{-1}?
3. The sounding of an atmosphere between 850 and 700 mb is represented by the points:

p(mb)	T(°C)
850	8.0
800	3.0
750	4.0
700	1.0

Plot these data on a tephigram, and compute the thickness of the layer by the methods of the mean isotherm and the mean adiabat.

4. The layer between 1000 and 900 mb has a constant lapse rate, with $+3\,^{\circ}\mathrm{C}$ at the base and $-3\,^{\circ}\mathrm{C}$ at the top. Derive its geopotential thickness by several approximation methods, using a tephigram, and express it in gpm, in dyn-m and in J kg^{-1}.
5. What is the height where the horizontal pressure gradient vanishes, when the pressure gradient at the Earth's surface, where $\bar{p} = 1000$ mb, is 0.011 mb km^{-1} and the temperature gradient at the same level 0.025 K km^{-1}? The pressure gradient and the temperature gradient are directed in opposite directions, and the horizontal temperature gradient should be considered as constant with height. The average temperature in the vertical is $\bar{T} = 272$ K. Treat the temperatures as virtual temperatures. (Hint: consider the layer thickness from the surface to the requested height.)