CHAPTER VII

THERMODYNAMIC PROCESSES IN THE ATMOSPHERE

In this chapter we shall study a number of thermodynamic processes of great importance in the atmosphere.

Two of these processes will be isobaric cooling (Sections 1 and 2). We have in that case

$$\delta q \neq 0; \qquad \mathrm{d}p = 0; \qquad \mathrm{d}h = \delta q \,.$$

All the other processes to be analyzed will be adiabatic, except that of Section 11. Three of them will also be isobaric; for them:

$$\delta q = 0; \qquad \mathrm{d}p = 0; \qquad \mathrm{d}h = \delta q + v\,\mathrm{d}p = 0 \,.$$

That is, they will be isenthalpic. And they will not in general be reversible. These processes are the vaporization or condensation of water in non-saturated air (Section 3), the horizontal mixing of two air masses (Sections 4 and 5) and the freezing of a cloud (Section 10). The vertical mixing of air masses (Section 12) may be considered as a combination of two adiabatic processes, one of them isobaric. Of the remaining processes, one will be non-adiabatic: the polytropic expansion (Section 11); another will be adiabatic and reversible, and therefore isentropic: the reversible expansion of saturated air (Sections 6 – 8). The last one will be performed with an open system: the pseudoadiabatic expansion (Section 9).

It is easy to understand why most of the important processes in the atmosphere are adiabatic. Our systems will in general be rather large portions of the atmosphere. We can generally disregard what happens on the ground surface and the cooling by radiation, and the heat conduction processes through the air are relatively inefficient. If the air parcel is large enough to be insensitive to what happens on its borders, we may consider it as a closed system which does not exchange heat with its surroundings.

7.1. Isobaric Cooling. Dew and Frost Points

In every closed system consisting of moist air, the specific humidity and the mixing ratio remain constant. This is not so for the partial pressure of water vapor or for the relative humidity. The first one is constant with temperature but proportional to the total pressure (cf. Chapter IV, Equations (76); also by considering $e = N_v p$). The relative humidity varies strongly with the temperature, due to the rapid variation of the saturation vapor pressure e_w, and is proportional to the pressure, through e.

Let us consider a mass of moist air cooling at constant pressure, and therefore contracting. The variables q, r and e will remain constant, but U_w will increase due to the decrease of e_w. If the cooling continues, e_w will become equal to e, and U_w will reach unity: the air has reached saturation. The temperature at which saturation is reached is called the *dew point*, T_d. This is a new variable that can be used to characterize the humidity of the air. If the saturation is reached with respect to ice, rather than to water (at a temperature below 0 °C), the temperature is called the *frost point*, T_f.

Now, let us assume that the pressure may change (by rising or subsiding of the air) and that the humidity of our system may also change (for instance, by incorporating water vapor by turbulent diffusion from a water surface, or because rain fell through the air mass), or that we simply want to compare air masses that differ in humidity and pressure. If we want to find the relation between the dew point and the mixing ratio and total pressure, we must apply the Clausius-Clapeyron equation describing the equilibirum curve water-vapor. We first differentiate logarithmically the approximate expression $e \cong pr/\varepsilon$ and obtain

$$\mathrm{d}\ln e \cong \mathrm{d}\ln r + \mathrm{d}\ln p\,. \tag{1}$$

The Clausius-Clapeyron equation is

$$\frac{\mathrm{d}\ln e}{\mathrm{d}T_d} = \frac{l_v}{R_v T_d^2} \tag{2}$$

where e is the vapor pressure in our air mass at temperature T. T_d is the dew temperature, and therefore corresponds to e over the saturation curve (see Figure VII-1). Due to this relation, T_d and e are humidity parameters giving equivalent information. Solving for $\mathrm{d}T_d$, we find

$$\mathrm{d}T_d = \frac{R_v T_d^2}{l_v}\,\mathrm{d}\ln e \cong \frac{R_v T_d^2}{l_v}\,(\mathrm{d}\ln r + \mathrm{d}\ln p)\,. \tag{3}$$

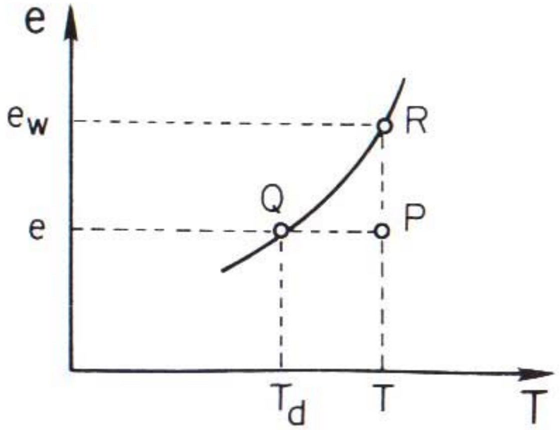

Fig. VII-1. Relation between temperatures and vapor pressures.

Or, if we want to express it as a relative variation $\mathrm{d}T_\mathrm{d}/T_\mathrm{d} = \mathrm{d}\ln T_\mathrm{d}$:

$$\mathrm{d}\ln T_\mathrm{d} = \frac{R_\mathrm{v}T_\mathrm{d}}{l_\mathrm{v}}\,\mathrm{d}\ln e \cong \frac{R_\mathrm{v}T_\mathrm{d}}{l_\mathrm{v}}(\mathrm{d}\ln r + \mathrm{d}\ln p) \cong 5\times 10^{-2}(\mathrm{d}\ln r + \mathrm{d}\ln p) \tag{4}$$

where the last expression corresponds to the approximation $T_\mathrm{d} \sim 270\,\mathrm{K}$ and indicates that the relative increase in T_d is about 5% of the sum of relative increases in r and p.

By integrating Equation (2) between T_d and T, we obtain

$$\ln\frac{e_\mathrm{w}}{e} = -\ln U_\mathrm{w} = \frac{1}{R_\mathrm{v}}\int_{T_\mathrm{d}}^{T}\frac{l_\mathrm{v}}{T^2}\,\mathrm{d}T \cong \frac{l_\mathrm{v}}{R_\mathrm{v}}\,\frac{T - T_\mathrm{d}}{TT_\mathrm{d}} \tag{5}$$

and solving for $(T - T_\mathrm{d})$, using decimal logarithms and substituting numerical values for the constants, we have

$$T - T_\mathrm{d} = 4.25\times 10^{-4}\,TT_\mathrm{d}(-\log U_\mathrm{w}), \tag{6}$$

which, for a rough estimate of $(T - T_\mathrm{d})$ as a function of U_w, can be written, with $TT_\mathrm{d} \cong 290^2$:

$$T - T_\mathrm{d} \cong 35(-\log U_\mathrm{w}). \tag{7}$$

Figure VII-1 shows the relations between the values for temperature and vapor pressure. The above integration was performed between Q and R.

So far we have considered the temperatures at which the equilibrium curve of phase transition is reached, such as T_d for point Q in Figure VII-1 or point D in Figure VII-2, and T_f for point F in Figure VII-2. But no thermodynamic argument can

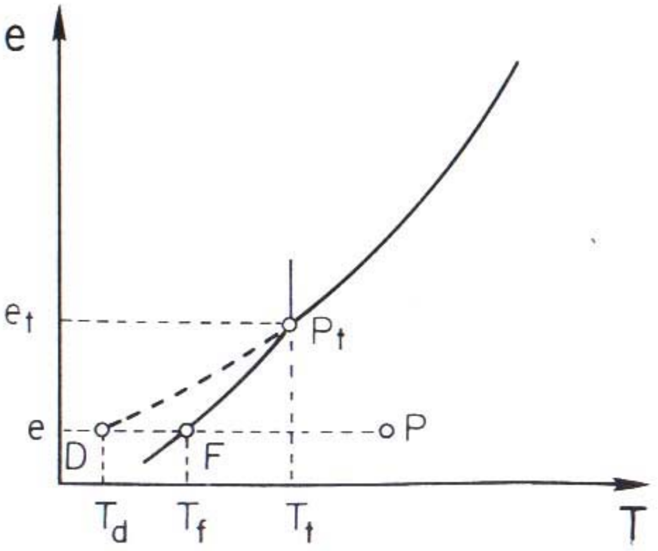

Fig. VII-2. Dew-and frost-points.

say whether the processes of condensation or sublimation will actually start taking place at these points, when isobaric cooling proceeds. Condensation does start at the dew point, although this process requires the presence of solid surfaces or of certain particles, called condensation nuclei. If neither of them were present, there would not be condensation (at least not along a certain temperature interval below the dew point) and the air would become *supersaturated* in water vapor (with $e > e_w$). However, atmospheric air always contains abundant condensation nuclei, and supersaturation does not occur to any appreciable extent. Therefore, although this is an important subject in the study of cloud physics, we do not need to be concerned with it here.

The situation is somewhat different regarding the processes of sublimation and of freezing. If ice surfaces are already present, sublimation or freezing will proceed readily on these surfaces as soon as the water or the vapor, as the case may be, reaches the equilibrium curve. In the absence of ice surfaces, on the other hand, although certain surfaces, either in macroscopic extensions or on minute particles of the atmospheric aerosol ("ice nuclei") favor the appearance of ice crystals, they only become active at temperatures well below the equilibrium curve. Neither does spontaneous nucleation take place with small supercooling of water or supersaturation of water vapor.

Therefore, when isobaric cooling of moist air proceeds, starting from a point such as P in Figure VII-2, sublimation will not in general occur at F. Between F and D, air will be supersaturated with respect to ice, and may only condense to water at the point D. It is interesting to derive the relation between dew and frost temperatures. This is an important point with respect to aircraft icing and surface fogs.

If we apply the Clausius-Clapeyron Equation (Chapter IV, Equation (50)) to both the vaporization and the sublimation curves, between the points D and F and the triple point P_t, we obtain:

$$\ln \frac{e_t}{e} = \frac{l_v}{R_v} \frac{T_t - T_d}{T_t T_d} = \frac{l_s}{R_v} \frac{T_t - T_f}{T_t T_f} \tag{8}$$

where e_t is the triple-point vapor pressure, and l_v and l_s are considered as constants. Taking into account that $T_t = 273.16\,\mathrm{K} \cong T_0 = 273.15\,\mathrm{K}$, that the dew and frost points, expressed in °C are

$$t_d = T_d - T_0 \cong T_d - T_t \tag{9}$$

$$t_f = T_f - T_0 \cong T_f - T_t \tag{10}$$

and that

$$T_t T_d \cong T_t T_f \tag{11}$$

we may write

$$\frac{t_d}{t_f} \cong \frac{l_s}{l_v}. \tag{12}$$

Although the mean values to be taken for the latent heats vary somewhat with the temperature interval, we may use with good approximation the values for −10 °C, which give

$$l_s/l_v \cong 9/8 . \tag{13}$$

From this value, it is easily seen that

$$|t_d - t_f| \cong |t_d|/9 \cong |t_f|/8 . \tag{14}$$

The dew point temperature can be found in a diagram by following the isobar from the image point P representing the air until it intersects the saturation mixing ratio line corresponding to the mixing ratio r of the air, as shown in Figure VII-3.

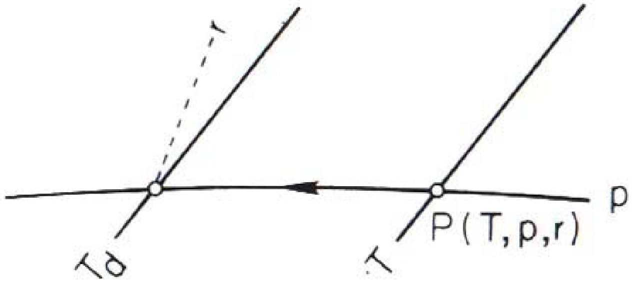

Fig. VII-3. Dew-point on a diagram.

We finally remark that, according to their definitions, T_d and T_f are (as e) invariants for isobaric changes of temperature of a closed system consisting of moist air.

7.2. Condensation in the Atmosphere by Isobaric Cooling

Dew and *frost* form as a result of condensation or sublimation of water vapor on solid surfaces on the ground, which cool during the night, by radiation, to temperatures below the dew or the frost point of the air in contact with them.

If a mass of atmospheric air cools isobarically until its temperature falls below the dew point, condensation will occur as microscopic droplets formed on condensation nuclei; we call this a *fog*. This occurs in the atmosphere due to the radiative cooling of the air itself or of the ground with which it is in contact (*radiation fogs*). As the droplets form, and because these droplets behave as black bodies in the wavelengths in which they irradiate, the radiation emitted by the layer increases, which favors further loss of heat. Condensation may also occur when an air mass moves horizontally over the ground toward colder regions, and becomes colder itself by heat conduction to the ground (*advection fogs*). In both cases, the cooling is practically isobaric, since pressure variations at the surface are usually very small (on a relative basis).

Once condensation starts, the temperature drops much more slowly, because the heat loss is partially compensated by the release of the latent heat of condensation. This sets a virtual limit very close to the dew point, an important fact in forescasting minimum temperatures.

For an isobaric process, the heat absorbed is given by the increase in enthalpy (Chapter IV, Equation (106)):

$$\delta q = \mathrm{d}h \cong c_\mathrm{p}\,\mathrm{d}T + l_\mathrm{v}\,\mathrm{d}r\,. \tag{15}$$

If we write $r \cong \varepsilon e/p$, as p is a constant, $\mathrm{d}r \cong (\varepsilon/p)\,\mathrm{d}e$. In this case e corresponds to saturation, and we may apply the Clausius-Clapeyron Equation (Chapter IV, Equation (48)) and write $e = e_\mathrm{w}$; we obtain:

$$\mathrm{d}r \cong \frac{\varepsilon}{p}\,\mathrm{d}e = \frac{\varepsilon l_\mathrm{v} e_\mathrm{w}}{pR_\mathrm{v}T^2}\,\mathrm{d}T \tag{16}$$

and

$$\delta q = \left(c_\mathrm{p} + \frac{\varepsilon l_\mathrm{v}^2 e_\mathrm{w}}{pR_\mathrm{v}T^2}\right)\mathrm{d}T\,, \tag{17}$$

or else

$$\delta q = \left(\frac{c_\mathrm{p} R_\mathrm{v} T^2}{l_\mathrm{v} e_\mathrm{w}} + \frac{\varepsilon l_\mathrm{v}}{p}\right)\mathrm{d}e_\mathrm{w}\,. \tag{18}$$

The relation between $\mathrm{d}T$ and $\mathrm{d}e_\mathrm{w}$ is indicated in Figure VII-4, on a vapor pressure diagram.

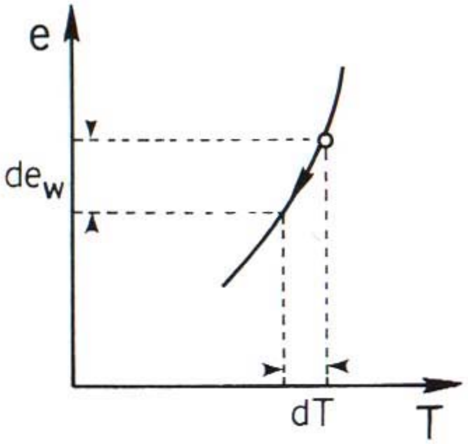

Fig. VII-4. Relation between the changes in temperature and in vapor pressure during condensation.

If we compute the heat loss $-\delta q$ from other data (e.g.: radiation loss), Equation (17) allows an estimation of the corresponding decrease in temperature $-\mathrm{d}T$. Similarly, the decrease in vapor pressure $-\mathrm{d}e_\mathrm{w}$ may be computed from Equation (18). From the gas law, the mass of water vapor per unit volume is given by $e_\mathrm{w}/R_\mathrm{v}T$, and its variation with temperature is $\mathrm{d}e_\mathrm{w}/R_\mathrm{v}T - (e_\mathrm{w}/R_\mathrm{v}T^2)\,\mathrm{d}T$. Introducing the Clausius-Clapeyron relation between de_w and dT, it can be shown that the second term is much smaller than the first one ($\sim 5\%$) and can be neglected for the following approximate argument. We can write thus for the mass of condensed water per unit volume dc:

$$dc \cong -\frac{1}{R_v T}\,de_w = -\frac{l_v e_w}{R_v^2 T^3}\,dT \tag{19}$$

(where the last relation is given, as before, by the Clausius-Clapeyron equation). If, for instance, we want to know the necessary cooling to reach a concentration of liquid water $\Delta c = 1$ g water m^{-3} air, starting with saturated air at 10°C, Equation (19) gives a result of $\Delta T = -1.6$°C (where we substituted finite differences for differentials).

We may notice that according to Equation (19), and using Δ instead of d for the variations,

$$-\Delta e_w \cong R_v T \Delta c \tag{20}$$

is roughly constant for a constant Δc in the usual interval of temperatures. Computation gives 1.3 to 1.4 mb for $\Delta c = 1$ g water m^{-3} air. We might use the vapor pressure diagram T, e to represent lines of equal Δc. Their points would lie Δe_w above the saturation curve for each temperature, as shown schematically in Figure VII-5. These lines of constant liquid water content in fogs are related to visibility. Equation (18) shows that *for a given loss of heat* $\delta q < 0$, $-de_w$ is larger if T is higher, because e_w within the bracket increases more rapidly than T^2; dc, approximately proportional to $(-de_w)$ according to (19), is therefore larger for higher T and smaller for lower T. For this reason dense fogs are less frequent at low than at mild temperatures.

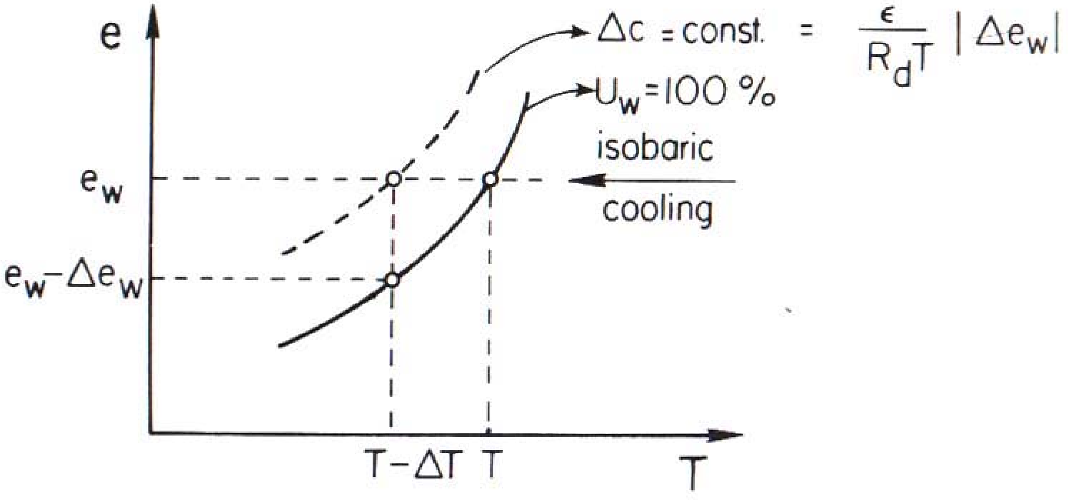

Fig. VII-5. Lines of constant liquid water content on a vapor pressure-temperature diagram.

7.3. Adiabatic Isobaric (Isenthalpic) Processes. Equivalent and Wet-Bulb Temperatures

We shall consider a closed system consisting of moist air and water. The same theory will hold for moist air and ice, with the corresponding substitutions (mass and specific heat of ice for mass and specific heat of water, latent heat of sublimation for latent heat of vaporization, etc.).

We shall study, for this system, adiabatic isobaric processes, therefore isenthalpic processes. We are interested in the expression for the enthalpy variation, which must be set equal to zero, thus providing an equation relating the temperature and humidity variables for these processes. The expression for enthalpy was calculated in Chapter IV, Section 14 under several forms; we shall use here Equation (104):

$$H = (m_d c_{pd} + m_t c_w)T + l_v(T) m_v + \text{const.} \tag{21}$$

If we consider two states of the system linked by an isenthalpic process, the above expression may be applied to one of them, and a similar expression to the other:

$$h' = (m_d c_{p_d} + m_t c_w)T' + l_v(T') m_v' + \text{const.}$$

where m_d, m_t and the additive constant are the same as for the former state. And from $h = h'$ we obtain:

$$(m_d c_{p_d} + m_t c_w)(T' - T) + l_v(T') m_v' - l(T) m_v = 0 \tag{22}$$

which may be written:

$$T' + \frac{l_v(T') m_v'}{m_d c_{p_d} + m_t c_w} = T + \frac{l_v(T) m_v}{m_d c_{p_d} + m_t c_w} . \tag{23}$$

The denominator on each side is a constant for each system. Each of the two sides of the equation is therefore a function of the state of the system only; in other words, this expression (either on the left or on the right side of Equation (23)) is an invariant for isenthalpic transformations. If we divide both numerator and denominator in the quotients by m_d, we have

$$T' + \frac{l_v(T') r'}{c_{p_d} + r_t c_w} = T + \frac{l_v(T)\, r}{c_{p_d} + r_t c_w} \tag{24}$$

where $r_t = (m_v + m_w)/m_d = m_t/m_d$. Notice that $(c_{pd} + r_t c_w)$, constant for each system, will vary for different systems according to their total contents in water substance (vapor plus liquid).

We may now start simplifying the expression by neglecting the heat capacity of the water:

$$T + \frac{l_v(T)\, r}{c_{p_d} + r c_w} \cong \text{const.}$$

Here the denominator is no longer a constant. If we also consider that rc_w is small in comparison with c_{pd}, and neglect the temperature variation of l_{v}, Equation (24) becomes

$$T' + \frac{l_{\mathrm{v}}}{c_{\mathrm{p}}} r' \cong T + \frac{l_{\mathrm{v}}}{c_{\mathrm{p}}} r \cong \text{const.} \tag{25}$$

where c_{p} may be taken as c_{pd} or, with better approximation, as $c_{\mathrm{pd}} + \bar{r} c_{\mathrm{w}}$, where $\bar{r}$ is an average value of r (an even better approximation, from Equation (24), would be to use the maximum value of r, with all water as vapor).

Equations (23) and (24) do not imply any more approximations than those made in deriving formula (104) from Chapter IV, Section 14. Equation (25) could also have been derived directly from the approximate Equation (106) in that chapter.

Let us consider the physical process that links two specific states (T, r) – unsaturated moist air plus water – and (T', r') – saturated or unsaturated moist air without water – of the system, and to which Equation (24) corresponds. We have 1 kg of dry air with $10^3\, r$ grammes of water vapor and $10^3\, (r' - r)$ grammes of liquid water (which may or may not be as droplets in suspension). We are thus assuming that $r' > r$, and also that saturation is not reached at any time (except eventually when arriving at the final state). The liquid water evaporates, and so the mixing ratio increases from r to r'. As this water evaporates, it absorbs the vaporization heat, which must be provided by the moist air itself and the water, because we are assuming that the system is adiabatically isolated. The temperature decreases from T to T'. As at any instant the state of the system differs finitely from saturation, the process is a spontaneous and irreversible one. A process producing the opposite modification (condensation of water) could be imagined, but it would be actually impossible. This is however immaterial in our case, because at no stage have we made the assumption of reversibility and only the first law has been applied. We shall deal with such a process first.

The *isobaric equivalent temperature* or, more simply, the *equivalent temperature* is defined as the temperature that moist air would reach if it were completely dried by condensation of all its water vapor, the water being withdrawn in a continuous fashion: the whole process is performed at constant pressure and the system is thermally isolated (except for the removal of water). We shall designate it by T_{ie} or T_{e}. With this definition, the process previously described and Equation (24) will not be strictly applicable, due to the removal of liquid. If we want to make an exact formulation, we should start by considering the infinitesimal variation dH:

$$(m_{\mathrm{d}} c_{\mathrm{pd}} + m_{\mathrm{v}} c_{\mathrm{w}})\, \mathrm{d}T + l_{\mathrm{v}}(T)\, \mathrm{d}m_{\mathrm{v}} = 0 \tag{26}$$

where we assume that there is no liquid water initially. $\mathrm{d}m_{\mathrm{v}}$ is assumed to be negative, which means that a mass $|\mathrm{d}m_{\mathrm{v}}|$ of vapor condenses. Now, before considering further condensation, we remove that liquid water. The enthalpy of the system will thereby decrease by $h_{\mathrm{w}}\, \mathrm{d}m_{\mathrm{v}}$, but this does not affect the value of T. For the next infinitesimal condensation, Equation (26) will again be valid, with the new value of m_{v}. Thus, the equation adequately describes the process; m_{v} and, therefore, the whole bracket in the first term are variable. An exact integrated expression could be obtained from (26) (see Problem 3). Within the usual approximation adopted in Equation (25), however, the term $m_v c_w$ is considered negligible, and we can write

$$T_{ie} = T + \frac{l_v}{c_{pd}} r \tag{27}$$

and introducing numerical values for l_v and c_{pd}:

$$T_{ie} = T + 2.5(10^3 r) \tag{28}$$

where $(10^3 r)$ is the mixing ratio expressed as (g of vapor kg^{-1} of dry air).

We may now consider the case when we go in a similar way from any value r to saturation, r_w. This is the process with which we are concerned in the use of the wet-bulb psychrometer. This instrument consists of two thermometers, one with a dry bulb to measure the air temperature, and the other one having its bulb covered with a wet muslin wick. Air must flow around the wet bulb and if it is non-saturated, water will evaporate until it becomes saturated. If steady state has been reached, the necessary enthalpy cannot come from the water, which has a constant temperature, but from the air itself. The thermodynamic system to be considered is a certain (any) mass of air that has flowed around the bulb, plus the mass of water which was incorporated into the air by evaporation from the muslin. We are thus considering a process similar to that which we have been studying, but with a difference: water has not undergone the same temperature variation as the initial air, as in the process of Equation (24), neither has it been incorporated and evaporated into the system at a variable temperature; in our present process the added water was from the beginning at the final temperature of the whole system. Again this difference is ignored in the approximation of Equation (25). We may then write, with the same approximation:

$$T_{ie} = T_{iw} + \frac{l_v}{c_{pd}} r_w = T + \frac{l_v}{c_{pd}} r, \tag{29}$$

where T_{iw} or T_w is the *isobaric wet-bulb temperature* or simply the *wet-bulb temperature*, and r_w is the saturation mixing ratio at the temperature T_{iw}. The wet-bulb temperature may thus be defined as the temperature which air attains when water is evaporated into it until saturation is reached, while the system (air plus water) is kept at constant pressure and does not exchange heat with the environment.

When Equation (29) is applied to compute T_{iw}, successive approximations must be made, as r_w is also unknown. In normal meteorological practice, however, the value of the isobaric wet-bulb temperature is known, and one wishes to deduce either the dew point, T_d, or the vapor pressure of the air, e, which is the saturation vapor pressure corresponding to the dew-point temperature. Replacing $r \cong \varepsilon e/p$ in (29) (right-hand equation), solving for e and considering that $e = e_w(T_d)$ (cf. Figure VII-1), we find

$$e_w(T_d) = e_w(T_{iw}) - \frac{c_{pd} p}{\varepsilon l_v}(T - T_{iw}). \tag{30}$$

This is one version of the so-called Psychrometric Equation. The coefficient of $p(T-T_{iw})$ is often referred to as the psychrometric constant, although it will vary slightly with T_{iw} (via the temperature dependence of l_v, and as a consequence of various second-order terms which have been ignored in Equation (30)).* At temperatures (T or T_{iw}) less than 0 °C, this equation still applies as long as the conventional wetted muslin evaporator does not freeze.

If we have ice (usually a very thin layer frozen on the thermometer bulb) as the evaporating phase, the equilibrium temperature is known as the *isobaric ice-bulb temperature*, T_{ii}. In this case, analogous to Equation (30),

$$e_w(T_d) = e_i(T_{ii}) - \frac{c_{pd}\,p}{\varepsilon l_s}(T - T_{ii}) = e_i(T_f), \tag{31}$$

where the actual vapor pressure can be interpreted in terms of either a dew-point or frost-point temperature (although the former is normally employed, as a conventional humidity parameter, in order to avoid ambiguity and apparent discontinuities in the moisture field).

The previous theory of the wet-bulb psychrometer does not take into account the possibility that part of the air flowing around the bulb may not reach saturation, or that the temperatures of air and water may not reach equilibrium. The identification of T_{iw} as defined by Equation (29) with the temperature read in the psychrometer is subject to experimental verification; this turns out to be satisfactory, provided there is enough ventilation**, and that the bulbs are small enough to exchange little infrared radiation with the screen or other objects at air temperature.

In regard to the process defining T_{iw}, the same observations as made for T_{ie} are pertinent. We deal with a process which is spontaneous (irreversible) in the sense of water evaporation; it would be an impossible one in the opposite sense.

Another important meteorological process is described by this type of transformation: air cooling by evaporation of rain, at a given level. In this case, the system consists of a certain air mass plus the water that evaporates into it from the rain that falls through it. The initial temperature of the water, in this case, will be that of the raindrops as they pass through that level, and the air may or may not reach saturation. If it does, its temperature will decrease to the value T_{iw}.

We may notice that T_{ie} and T_{iw} are linked by Equation (29), and they are the maximum and the minimum value, respectively, that the air may attain through the isenthalpic process that we have considered. They are therefore two parameters giving equivalent information about the temperature and humidity state of the air. This relation can be expressed graphically by representing the process on a vapor pressure

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* Its value can be expressed by 0.000660 $(1 + 0.00115\, t_{iw})$, where t_{iw} is the Celsius wet-bulb temperature.

** An air flow between 4 and 10 m s^{-1} over the bulbs is recommended.

diagram. If we substitute the approximate expression $\varepsilon e/p$ for r in Equation (25) and rearrange the latter, we obtain

$$e' - e = -\frac{c_{p_d} p}{\varepsilon l_v}(T' - T). \tag{32}$$

For each value of p, $(T' - T)$ is proportional to $(e' - e)$; i.e., the adiabatic isobaric process in which we are interested occurs in the diagram along a straight line passing through the image point $P(T, e)$ with a slope $-(c_{p_d}/\varepsilon l_v)p$. This is represented in Figure VII-6. Extending the line toward increasing temperatures, it will intersect the horizontal axis $(e = 0)$ at T_{ie}, and in the other direction it intersects the saturation curve at T_{iw}.

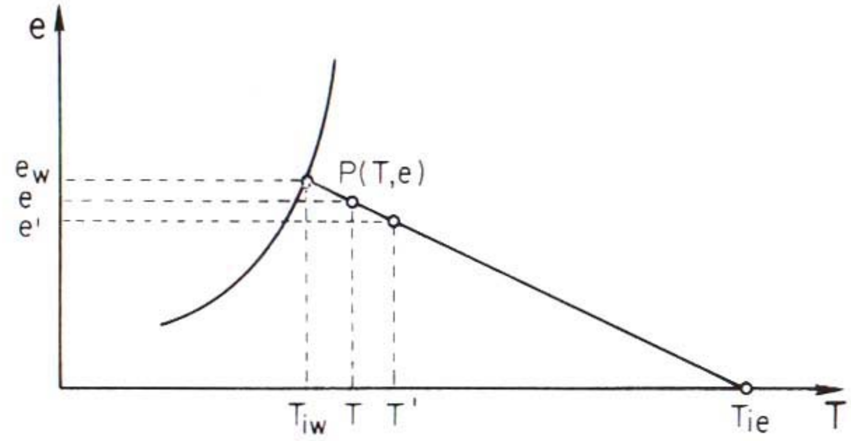

Fig. VII-6. Adiabatic isobaric process represented on a vapor pressure-temperature diagram.

7.4. Adiabatic Isobaric Mixing (Horizontal Mixing) Without Condensation

We shall now consider another adiabatic isobaric (and therefore isenthalpic) process: the mixing of two moist air masses, with different temperatures and humidities, but at the same pressure. This process corresponds in the atmosphere to horizontal mixing. Condensation is assumed not to take place.

If we use the subscripts 1 and 2 for the two masses, we shall have

$$m_1 h_1 + m_2 h_2 = (m_1 + m_2) h$$

or

$$\Delta H = m_1(h - h_1) + m_2(h - h_2) = m_1 \Delta h_1 + m_2 \Delta h_2 = 0, \tag{33}$$

where

$$\Delta h_1 = c_{p_1}(T - T_1)$$

$$\Delta h_2 = c_{p_2}(T - T_2);$$

T being the final temperature, and (Chapter IV, Equation (87))

$$c_{p_i} = c_{p_d}(1 + 0.87\,q_i)$$

($i=1, 2$). Substituting in ΔH:

$$m_1 c_{p_d}(1 + 0.87\,q_1)(T - T_1) + m_2 c_{p_d}(1 + 0.87\,q_2)(T - T_2) = 0\,. \tag{34}$$

And solving for T:

$$T = \frac{(m_1 T_1 + m_2 T_2) + 0.87(m_1 q_1 T_1 + m_2 q_2 T_2)}{m + 0.87(m_1 q_1 + m_2 q_2)}\,. \tag{35}$$

The total mass is $m = m_1 + m_2$, and the total mass of water vapor $m_1 q_1 + m_2 q_2$. As this must remain constant, it will be equal to mq, q being the final specific humidity. Therefore q is the weighted average of q_1 and q_2:

$$q = \frac{m_1 q_1 + m_2 q_2}{m}\,. \tag{36}$$

Introducing this relation in Equation (35) gives

$$T = \frac{(m_1 T_1 + m_2 T_2) + 0.87(m_1 q_1 T_1 + m_2 q_2 T_2)}{m(1 + 0.87\,q)}\,.$$

If we neglect the water vapor terms, Equation (35) becomes:

$$T \cong \frac{m_1 T_1 + m_2 T_2}{m}\,. \tag{37}$$

That is, the final temperature is approximately given by the weighted average of the initial temperatures.

The potential temperature θ of the mixture is also given by similar formulas, as may be seen by multiplying both sides of Equation (35) or (37) by $(1000/p)^{\varkappa}$, taking into account that p is a constant. In particular, we conclude that θ is also given approximately by the weighted average of θ_1 and θ_2:

$$\theta \cong \frac{m_1 \theta_1 + m_2 \theta_2}{m}\,. \tag{38}$$

If we use the approximate relation between e and q ($q \cong \varepsilon e/p$; see Chapter IV, Section 11) we obtain for the final vapor pressure:

$$e \cong \frac{m_1 e_1 + m_2 e_2}{m}\,. \tag{39}$$

We may notice that Equation (36) is valid for any mixture of two air masses, without condensation, independent of the pressure values or variations, while Equation (39) is only valid for an isobaric mixture, because in deriving it we assumed $p_1 = p_2 = p$.

Formulas (36), (37) and (39) indicate that the final q, T and e are obtained by computing the averages of their initial values, weighted with respect to the masses. Thus if P_1 and P_2 are the image points of the two air masses in a vapor pressure diagram (Figure VII-7), the image point P of the mixture will lie on the straight line joining P_1 and P_2, at a distance such that $P_1P/PP_2 = m_2/m_1$ *.

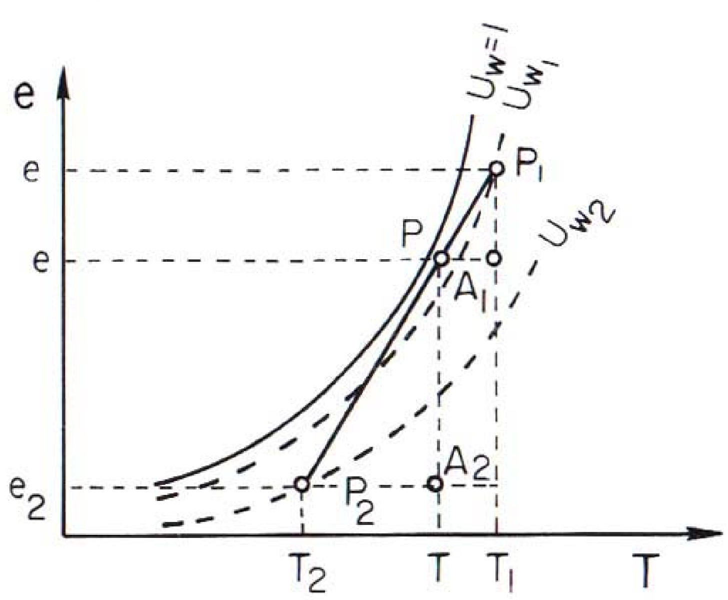

Fig. VII-7. Effect of mixing on relative humidity.

The dashed curves are curves of equal relative humidity ($U_{w1} = e_1/e_{w1}$ and $U_{w2} = e_2/e_{w2}$, respectively). It is easy to see that, due to the curvature of the relative humidity isopleths, the value of U_w of the mixture will always be higher than the weighted average of U_{w1} and U_{w2}; for instance, if $U_{w1} = U_{w2} = U_w$, any mixture will give a point to the left of the curve U_w.

7.5. Adiabatic Isobaric Mixing With Condensation

It becomes obvious on the vapor pressure diagram that, due to the increase in relative humidity produced by mixing, we may have the case of Figure VII-8, where the image point of the resulting mixture corresponds to a state of supersaturation, although both initial air masses were unsaturated.

As appreciable supersaturations cannot be realized in atmospheric air, condensation of water into droplets will occur in this case. Independently of how the real process occurs in nature, if we take into account that enthalpy changes do not depend on the

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* This is easily seen by writing Equation (37) in the form

$$\frac{T - T_1}{T_2 - T} = \frac{m_2}{m_1}$$

and Equation (39) in a similar form, and considering the similarity of triangles PP_1A_1 and P_2PA_2 in Figure VII-7.

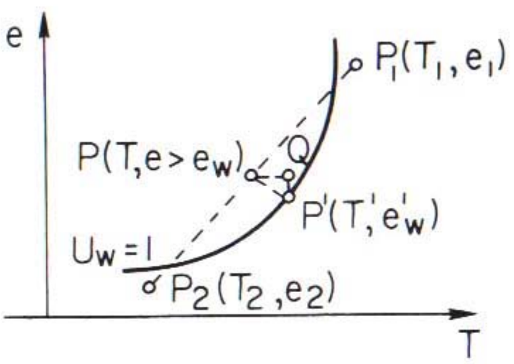

Fig. VII-8. Condensation produced by mixing.

path, but only on the initial and final states, we may consider the whole process as consisting (1) of a mixture giving supersaturated air as represented by P, followed by (2) condensation of water until the saturation vapor pressure is reached. The latter is the same type of isenthalpic process studied in Section 3, and can be represented on the diagram by the corresponding straight line PP'. The latent heat released is responsible for the heating from T to T'.

It may be noticed that the amount of liquid water produced per unit mass of air is given by the difference $q - q'_w \cong r - r'_w \cong (\varepsilon/p)(e - e'_w)$, where $(e - e'_w)$ is represented on the diagram by the segment $P'Q$. The corresponding concentration of liquid water per unit volume (cf. Section 2, Equation (20)) will be given by $\Delta c = (e - e'_w)/R_v T$.

This mechanism can in principle produce fogs (*mixing fogs*), but the amounts of condensed phase produced by it in the atmosphere are very low. It is however efficient in assisting other mechanisms (such as radiative cooling) to produce fogs.

A particular kind of mixing fog is *steam fog*, produced when cold air moves over warmer water. The layer in contact with the water will be saturated at the water temperature. As the cold air moves over this layer, a mixing process takes place, resulting in condensation. Due to the temperature stratification, these fogs are unstable, vertical stirring dissipating them into the drier air above. An example of this type of fog is the so-called 'Arctic Sea smoke', produced by very cold air passing over open water, e.g., in the North Atlantic or Baffin Bay or in open leads in the Arctic Ocean.

A further important example of adiabatic isobaric mixing processes occurs during the combustion of fuels when the exhaust gases are mixed with the atmosphere. When temperatures are low, fog may result from natural sources of combustion products and at upper levels condensation trails may be produced by aircraft. Since such effects are very important operationally, it is of interest to derive criteria for the occurrence of such phenomena. We may assume that condensation will be initiated, regardless of the temperature, only if the air in a mixed volume becomes saturated with respect to liquid water. At those temperatures for which this generally occurs the freezing nuclei content is usually adequate to cause transformation of the condensed phase to ice crystals, whose growth will continue until the vapor content is reduced to the ice-saturation value. The significance of the phenomenon will depend on the concentration of the condensed (ice) phase. For combustion processes at the surface, this concentration will depend on many factors – the rate of fuel consumption, the local topography, the intensity of vertical mixing, the horizontal wind speed, etc. In this case, a quantitative assessment of horizontal visibility (or of ice crystal concentration) is in general impossible. In the case of aircraft condensation trails, the situation is rather different since the generating element (the aircraft) is moving very rapidly relative to the air so that local accumulation of condensed phase is impossible. Accordingly, the aircraft problem will be analyzed in some detail, but it will be indicated how the primary criterion, for initiation of condensation, may be applied to surface combustion processes. We shall consider only a jet aircraft, for which all the heat produced by combustion is ejected into the exhaust gases.

We may imagine that the aircraft ejects heat and water vapor into a long thin cylinder along its flight path. The only significant diffusion and mixing will take place at right angles to this line so that one can visualize the resulting trail (visible or not) as consisting of a conical-shaped volume of air to the rear of the aircraft. The air along the central axis of this cone will most closely resemble the air in the original cyclindrical tube, and the air at the exterior limits will be essentially identical with undisturbed environmental air. We may assume that the heat and water vapor are diffused outwards in a similar manner and that radiative cooling plays a negligible role. For simplicity in numerical analysis, and because the geometry of the diffusion processes is not at all critical, we may imagine all sections normal to the cone axis to be completely mixed, over a cross section, A.

Let us first derive the conditions under which saturation with respect to liquid (or supercooled) water just occurs at a single value of A. Let us imagine that m_1 kg dry air plus $m_1 r$ kg water vapor are drawn into the jet engine, combined with F kg fuel (the fuel consumption per unit length of path) and exploded, adding FQ J of heat energy (Q is the heat of combustion, in J kg^{-1}, expressed in terms of gaseous products of combustion) and Fw kg water vapor to a volume A, containing originally m_2 kg dry air and $m_2 r$ kg water vapor.

Within the volume A, corresponding to a unit length of trail, the temperature will be in excess of that of the environment and will be denoted by $T+\Delta T$; the air will be assumed to be saturated with respect to liquid water and to contain a concentration E of the condensed phase (here assumed to be liquid). The First Principle of Thermodynamics enables us to state (neglecting the heat capacities of liquid water and combustion products other than water vapor; cf. Chapter IV, Equation (104), and notice that the heat term FQ must also be added in the present case)

$$FQ + l_v EA = (m_1 + m_2)\Delta T(c_{p_d} + rc_{p_v}). \tag{40}$$

The conservation of total water substance (r_t) requires that

$$(m_1 + m_2)r + Fw = (m_1 + m_2)r_w + EA = (m_1 + m_2)r_t. \tag{41}$$

The equation of state for the resulting gas phase is (cf. Chapter IV, Equation (70))

$$v = \frac{A}{(m_1 + m_2)(1 + r_w)} = \frac{R(T + \Delta T)}{p} = \frac{R_d(T + \Delta T)}{p}\left[1 + \left(\frac{1}{\varepsilon} - 1\right)r_w\right]. \tag{42}$$

Therefore

$$\frac{A}{m_1 + m_2} = \frac{R_d(T + \Delta T)}{p}\left(1 + \frac{r_w}{\varepsilon}\right). \tag{43}$$

Dividing Equation (40) by $(m_1 + m_2)$ and using Equation (43) gives

$$\Delta T\left[c_{p_d} - \frac{R_d}{Ap}(FQ + l_v EA)\right] = \frac{R_d T}{Ap}(FQ + l_v EA), \tag{44}$$

where we have ignored some trivial terms in r and r_w. Making the additional yet equivalent assumption that vapor pressures are small relative to the total pressure, a similar treatment of Equation (41) gives, making appropriate substitutions for r and r_w,

$$\frac{\varepsilon e_w}{p} U_w = \frac{\varepsilon(e_w + \Delta e_w)}{p} + \frac{R_d(T + \Delta T)}{Ap}(EA - Fw). \tag{45}$$

If ΔT at the point in the trail with maximum of trail density E is small relative to T, we can invoke the Clausius-Clapeyron equation in the form

$$\Delta e_w = \frac{\varepsilon l_v e_w}{R_d T^2} \Delta T. \tag{46}$$

Substituting Equations (44) and (46) into Equation (45), and collecting terms, gives

$$E\left[\frac{\varepsilon l_v e_w}{p}\left(\frac{\varepsilon l_v}{R_d T} + U_w - 1\right) + c_{p_d} T\right] + \frac{F}{A}\left[\frac{\varepsilon e_w Q}{p}\left(\frac{\varepsilon l_v}{R_d T} + U_w - 1\right) - c_{p_d} T w\right] + \frac{c_{p_d}}{R_d}\varepsilon e_w (1 - U_w) = 0. \tag{47}$$

The coefficient of E will always be positive (since $\varepsilon l_v \gg R_d T$), and the final term zero (for saturated air) or positive. The coefficient of F/A can be either positive or negative. If positive, E will be everywhere negative (trail everywhere unsaturated). If the coefficient of F/A is negative, the trail density will be positive, at least for small A (and a maximum for vanishing A). If the environmental air is not saturated, the trail density becomes zero for some value of A, and whether or not a trail forms will depend on conditions close behind the aircraft (where ΔT and Δe_w may well be large). If the environmental air is saturated with respect to the liquid phase, a single criterion for positive trail density exists for all values of A (hence is valid for large A and small Δe_w and small ΔT, for which the above derivation is reasonably accurate). Thus, for saturated air, condensation can occur if

$$c_{p_d} T w > \frac{\varepsilon^2 e_w l_v Q}{R_d p T} \tag{48}$$

or, at a fixed temperature, if the pressure exceeds a critical pressure

$$p_c = \frac{\varepsilon^2 e_w l_v}{c_{p_d} R_d T^2} \frac{Q}{w}. \tag{49}$$

Inserting reasonable values for the ratio Q/w (appropriate for kerosene or a similar fuel) reveals that trails can only form at very low temperatures, achieved in the Arctic in the winter at the ground and elsewhere in the upper troposphere when the tropopause is high and cold. Since the air under these conditions would seldom, if ever, be saturated with respect to liquid water, it is necessary to carry out a more elaborate analysis than that given above if universal criteria for condensation trails are desired.

Even if temperatures in the trail, near the aircraft, are of the order of 10 to 20 °C one can still assume $(e_w + \Delta e_w) \ll p$ but one can no longer treat e_w as a linear function of temperature. Let us, therefore, replace Equation (46) by a higher-order approximation. It is adequate to treat $l_v T^{-2}$ as virtually constant, so that

$$\Delta e_w \cong \frac{\varepsilon l_v e_w}{R_d T^2} \Delta T + \frac{e_w}{2} \left(\frac{\varepsilon l_v \Delta T}{R_d T^2} \right)^2. \tag{50}$$

Correct to this order of accuracy, one can use in the final term in Equation (47) the approximate relation, obtained by neglecting ΔT in Equation (43) when substituting in Equation (40),

$$(\Delta T)^2 \cong \frac{R_d T}{A p c_{p_d}} (FQ + l_v EA) \Delta T. \tag{51}$$

Substituting Equations (44), (50) and (51) into Equation (45), and collecting terms, gives

$$EB_1 + \frac{F}{A} B_2 + \left(\frac{F}{A} \right)^2 B_3 + B_4 = 0, \tag{52}$$

where B_1, B_2 and B_4 have the same values as they did in Equation (47), and where

$$B_3 = \frac{\varepsilon^3 e_w l_v^2 Q^2}{2 c_{p_d} R_d p^2 T^2}. \tag{53}$$

In the above derivation, the correction term has been obtained by maintaining only the first (Q^2) term in the expansion of $(FQ + l_v EA)^2$, since we are primarily interested in solutions for which E is very small and A is relatively small.

E will have a maximum value, E_m, for a value A_m of the trail cross-section such that $\partial E/\partial A$ is zero, yielding

$$\frac{F}{A_m} = - \frac{B_2}{2B_3} \tag{54}$$

$$E_m = \frac{B_2^2}{4B_1B_3} - \frac{B_4}{B_1}. \tag{55}$$

The critical conditions for a trail are those for which E_m is zero, or

$$B_2 + 2(B_3B_4)^{1/2} = 0. \tag{56}$$

Substituting for the values of the B parameters, the critical pressure, p_c, becomes

$$p_c = \frac{\varepsilon^2 e_w l_v}{c_{p_d} R_d T^2} \frac{Q}{w} \left\{ 1 + [2(1 - U_w)]^{1/2} - \frac{R_d T}{\varepsilon l_v}(1 - U_w) \right\}. \tag{57}$$

The final term inside the brackets can be ignored since $R_d T \ll \varepsilon l_v$. In can be seen that Equations (57) and (49) are identical for $U_w = 100\%$, but differ substantially if the air is relatively dry. This does not, however, imply a significant shift relative to temperature since the critical curves are steep when plotted on a $(T, \ln p)$ diagram. This can be demonstrated by logarithmic differentiation of Equation (57), assuming as before that e_w varies much more rapidly with temperature than does $l_v T^{-2}$. It follows (using the Clausius-Clapeyron equation) that

$$\left(\frac{\partial \ln p_c}{\partial T}\right)_{U_w} \cong \frac{\varepsilon l_v}{R_d T^2}, \quad \text{or} \quad \left(\frac{\partial \ln p_c}{\partial \ln T}\right)_{U_w} \ll 1. \tag{58}$$

Condensation trails will occur for $p > p_c(T)$ or for $T < T_c(p)$, as defined by Equation (57).

This situation is illustrated on Figure VII-9, in terms of a $(T, \ln p)$ diagram. Thus, in zone III, condensation trails are impossible since supersaturation relative to liquid water is not observed in the atmosphere. In zone I condensation trails will always

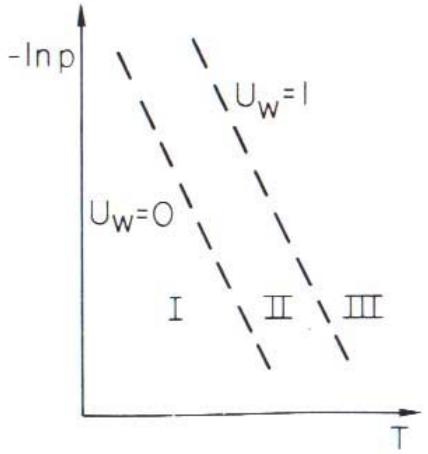

Fig. VII-9. Zones of an emagram relative to condensation trails.

form, regardless of the relative humidity. In zone II the formation of condensation trails, or their absence, will depend on the ambient relative humidity; formation will be possible if the (p, T) point lies to the left of the U_w line corresponding to the actual relative humidity.

Equation (57) can also be used to predict the formation of low temperature fog. It does not depend on the rate of fuel consumption but only on the ratio of heat to water vapor in the exhaust gases. For engines and furnaces of the normal type Q must be reduced by the amount abstracted for heating purposes or for the performance of work. Eventually, of course, such heat would reach the atmosphere (conduction through walls, frictional losses, etc.) so that if one does not deal with localized sources (a chimney or exhaust pipe) the full value of Q may be employed.

It is now important, at least for jet aircraft trails, to deduce the criteria for visible trails. It has been estimated that for a condensation trail to be visible from a distance the trail density (of ice phase), I in kg m^{-3}, must exceed 4×10^{-6} kg m^{-3}. This value will be taken as the critical trail density, I_c, and we shall investigate the conditions under which this value is just achieved at a single value of the cross-sectional area, A_c. This will correspond to a very small just-visible trail volume, moving along at a fixed distance behind the aircraft. For denser, longer and hence more-persistent trails, values of $I > I_c$ can be investigated. Fortunately, at a fixed p and r, temperatures only a few degrees colder than that yielding $I = I_c$ give a trail density very much greater than I_c.

We must now reformulate our basic equations to take into account the condensation, as ice particles, of I kg m^{-3}, and the resultant saturation of the air with respect to ice (provided of course that saturation relative to the liquid phase was achieved first). The equations are, of course, very similar to the previous set and can be written down immediately by analogy, viz.

$$IC_1 + \frac{F}{A} C_2 + \left(\frac{F}{A}\right)^2 C_3 + C_4 = 0, \tag{59}$$

where

$$C_1 = \frac{\varepsilon l_s e_i}{p}\left(\frac{\varepsilon l_s}{R_d T} + U_i - 1\right) + c_{p_d} T,$$

$$C_2 = \frac{\varepsilon e_i Q}{p}\left(\frac{\varepsilon l_s}{R_d T} + U_i - 1\right) - c_{p_d} T w, \tag{60}$$

$$C_3 = \frac{\varepsilon^3 e_i l_s^2 Q^2}{2 c_{p_d} R_d p^2 T^2} \quad \text{and} \quad C_4 = \frac{c_{p_d}}{R_d} \varepsilon e_i (1 - U_i).$$

By analogy to Equation (55), it follows that a visible trail is just possible if

$$I_m = I_c = \frac{C_2^2}{4 C_1 C_3} - \frac{C_4}{C_1}. \tag{61}$$

Substituting from the set in Equation (60), and recalling that $\varepsilon l_s \gg R_d T$, one obtains

$$1 - U_i = \frac{1}{2}\left(1 - \frac{c_{p_d} R_d p T^2}{\varepsilon^2 e_i l_s} \frac{w}{Q}\right)^2 - I_c\left(\frac{\varepsilon l_s^2}{c_{p_d} p T} + \frac{R_d T}{\varepsilon e_i}\right). \tag{62}$$

For consistency with criterion (57), which must also be obeyed, we can introduce U_w by

$$U_i e_i = U_w e_w . \tag{63}$$

The final criterion curves can be obtained by compounding the two critical curves (just saturation re liquid water and $I_{max} = I_c$) for a given U_w value, choosing the segment at any pressure giving the lower critical temperature. The above equation is best solved by computing U_w as a function of T(or p) for a fixed pressure (or temperature) and interpolating the critical temperature (or pressure) for chosen U_w values.

With the above results, we may also investigate the criterion for intensification or dissolution of cirrus cloud when a jet aircraft flies through such cloud, or the parallel case of using a burner to attempt to dissipate fog when surface temperatures are low. In this case $I_c = 0$ and $U_i = 100\%$, so that the critical pressure below which cirrus consumption will occur is, from Equation (62),

$$p_c' = \frac{\varepsilon^2 e_i l_s}{c_{p_d} R_d T^2} \frac{Q}{w}. \tag{64}$$

It may be noted that the converse effect (i.e., condensation in the ice phase) is often responsible for dense ice crystal fog at very low temperatures due to aircraft warm-up and take-off, at times sufficiently severe to cause a temporary cessation of flying.

7.6. Adiabatic Expansion in the Atmosphere

The processes of adiabatic expansion (or compression) are particularly important, because they describe the transformations taking place when an air mass rises (or descends) in the atmosphere. They are therefore a part of the study of convection, and we may consider what happens when a parcel of the atmosphere rises without mixing with its environment, that is, during the adiabatic expansion of moist air. In the more general case, the entire atmosphere is assumed to rise; the relations are, of course, identical.

The first stage will be a moist adiabatic expansion of only one gaseous phase; this is a simple process which was already considered in Chapter VI, Section 4, where we saw that it differs very little from a dry adiabatic expansion. In other words, we can write with good approximation $\varkappa \cong i_d$ (cf. Chapter IV, Equation (89)), and represent the process by the equation

$$\theta = T\left(\frac{1000}{p}\right)^{\varkappa_d} \tag{65}$$

(p in mb), which on a diagram corresponds to the dry adiabats; each curve is characterized by a potential temperature θ, which must be the value of T where the curve intersects the 1000 mb isobar. Along each of these curves, entropy has a constant value $s = c_{p_d} \ln \theta + \text{const.}$ These relations are shown in Figure VII-10 for a tephigram.

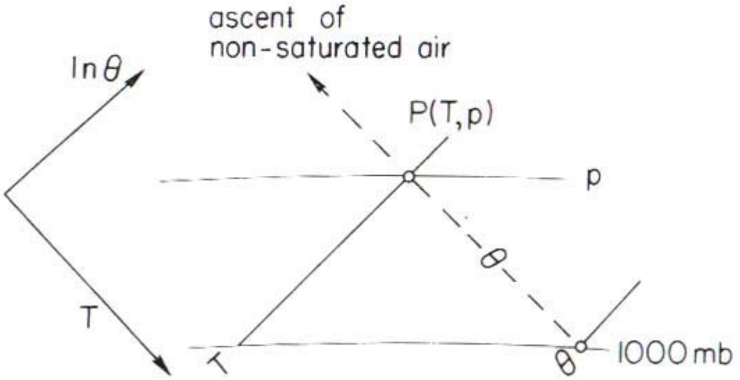

Fig. VII-10. Non-saturated adiabatic expansion on a tephigram.

The decrease in temperature will continue until saturation is reached, that is, until the temperature is such that the air humidity is that of saturation (r_w becomes equal to r). From that moment on, the second stage proceeds. In this stage two possibilities may be considered. The water (or ice) produced by condensation may remain in the air (cloud), so that if we reverse the process, the same water will evaporate during the adiabatic compression along the descent; in this case, we shall call it a *reversible saturated adiabatic expansion*. It is implied that the process must be slow enough to be considered reversible. Being adiabatic and reversible, it is also isentropic.

We may also assume that the water or ice falls out of the system as soon as it is produced. In this case we are dealing with an open system, and the process is called *pseudo-adiabatic*.

Often an intermediate process will operate in the atmosphere, where part (not all) of the water or ice condensed during the ascent falls out as precipitation.

Due to the condensation heat, cooling is slower in a saturated than in a moist expansion. But as condensation must proceed along with the cooling (r_w decreases), saturated adiabats must have, as was explained in Chapter VI, Section 4, an intermediate slope between moist adiabats and the saturation mixing ratio isopleths.

It has been customary to talk of a *rain stage*, a *snow stage*, and a *hail stage*. By the rain stage is meant the reversible adiabatic expansion during the ascent of air, with production of liquid water; the snow stage refers to the production of ice. Of course the temperature (and therefore the height in the atmosphere) determines whether we are in one or the other stage. The transition does not occur in general at 0 °C, but at lower temperatures. At air temperatures less than $-40\,^\circ\text{C}$, only sublimation occurs, because at those low temperatures liquid droplets freeze spontaneously, so that only ice clouds can exist. Mixed clouds can exist at intermediate temperatures between

0 °C and −40 °C, but obviously they are not in thermodynamic equilibrium; the ascent of the air will produce both condensation on droplets and sublimation on ice crystals, while a continuous process of distillation from the droplets (with higher vapor pressure) to the ice crystals (with lower vapor pressure) must occur. The so-called hail stage assumed that at the level with temperature 0 °C, the cloud water would freeze isentropically (reversible adiabatic freezing). Nothing of that kind ever occurs in the atmosphere, where freezing of droplets occurs only (at temperatures in excess of −40 °C) on *ice nuclei* (particles that favor freezing), and these only start being active at temperatures considerably below 0 °C, in increasing numbers as the temperature decreases; in these conditions the freezing is of course irreversible. Our analysis (in Section 10) will deal with the type of hail stage that actually occurs.

We shall first consider the attainment of saturation by adiabatic expansion of moist air (Section 7), and in later sections (Sections 8 and 9) the saturated expansion.

7.7. Saturation of Air by Adiabatic Ascent

In Section 1 we have dealt with the saturation by isobaric cooling. We shall now consider saturation due to adiabatic expansion by ascent in the atmosphere.

If we differentiate logarithmically the definition of U_w, we have

$$\mathrm{d}\ln U_w = \mathrm{d}\ln e - \mathrm{d}\ln e_w\,. \tag{66}$$

During the ascent $e/p = N_v$ is a constant. Therefore, from Poisson's equation

$$Tp^{-\varkappa} = \text{const.} \tag{67}$$

we derive

$$Te^{-\varkappa} = N_v^{-\varkappa}\,\text{const.} = \text{const.}'\,. \tag{68}$$

That is, the partial pressure also obeys Poisson's equation. Differentiating logarithmically:

$$\mathrm{d}\ln T = \varkappa\,\mathrm{d}\ln e\,. \tag{69}$$

Introducing this expression and Clausius-Clapeyron's equation, we obtain for the logarithmic variation of relative humidity:

$$\mathrm{d}\ln U_w = \frac{1}{\varkappa}\,\mathrm{d}\ln T - \frac{l_v}{R_v T^2}\,\mathrm{d}T\,. \tag{70}$$

Here the first term on the right hand side gives the change due to the decrease in pressure p (and therefore in e), while the second term measures the influence of the decrease in temperature and therefore in e_w and is of the same form as for an isobaric cooling (cf. Section 1). The two terms have opposite signs, so that in principle an adiabatic expansion could lead to a decrease as well as to an increase in U_w.

Equation (70) can also be written:

$$\frac{\mathrm{d}U_w}{\mathrm{d}T} = \frac{U_w}{T}\left(\frac{1}{\varkappa} - \frac{l_v}{R_v T}\right) = \frac{U_w}{T}\left(\frac{c_p T - \varepsilon l_v}{R_d T}\right). \tag{71}$$

The expression within brackets is <0 if

$$T < \varepsilon l_v/c_p \cong 1500\,\mathrm{K} \tag{72}$$

a condition which always holds in the atmosphere. Therefore U_w increases as T decreases, for an adiabatic expansion.

If we represent the initial state of the ascending air on a vapor pressure diagram by the image point P (see Figure VII-11), the arrow starting from P indicates the adiabatic ascent described by Equation (69). As the air temperature T changes, the saturation vapor pressure will simultaneously change according to the arrow starting from S, along the saturation curve, which is described by the Clausius-Clapeyron equation.

If we integrate Equation (70), we obtain

$$\ln \frac{U_w}{U_{w_0}} = \frac{1}{\varkappa} \ln \frac{T}{T_0} + \frac{l_v}{R_v}\left(\frac{1}{T} - \frac{1}{T_0}\right), \tag{73}$$

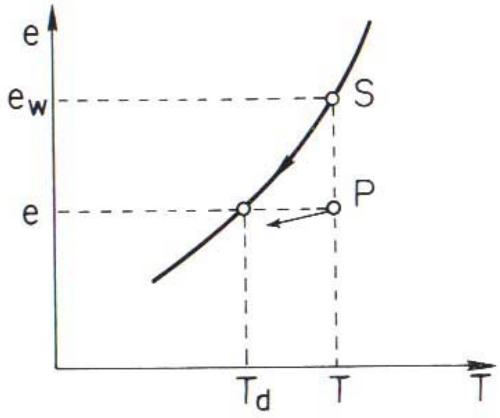

Fig. VII-11. Non-saturated adiabatic expansion on a vapor pressure-temperature diagram.

where T_0, U_{w_0} refer to initial conditions. For $U_w = 1$

$$-\ln U_{w_0} = \frac{1}{\varkappa} \ln \frac{T_s}{T_0} + \frac{l_v}{R_v}\left(\frac{1}{T_s} - \frac{1}{T_0}\right) \tag{74}$$

which can be solved numerically for the *saturation temperature* T_s. In the atmosphere the ascent can always lead to saturation, i.e., T_s can always be reached. On a tephigram, this means that starting from the image point and rising along the dry adiabat, we shall eventually reach the vapor line corresponding to the value r of the air (see Figure VII-12). On the vapor pressure diagram (Figure VII-11) it means that the adiabat starting from P will reach the saturation curve.

We shall see later that, along a dry adiabat, the temperature drops approximately 10°C for every km. We want now to make an estimation of the height at which

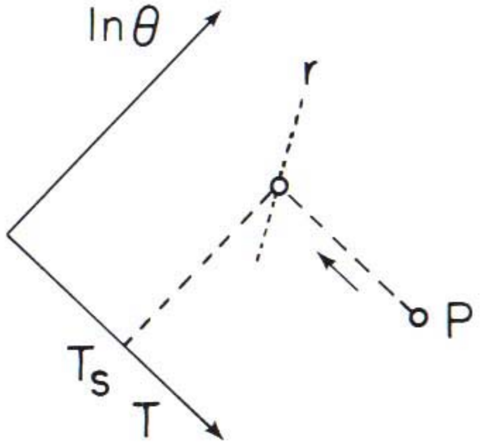

Fig. VII-12. Saturation by adiabatic expansion, on a tephigram.

saturation will be attained. For that purpose we shall first inquire what is the variation of the dew point temperature along the adiabat. This is given by the Clausius-Clapeyron equation (cf., Figure VII-11):

$$\mathrm{d}T_\mathrm{d} = \frac{R_\mathrm{v} T_\mathrm{d}^2}{l_\mathrm{v}}\,\mathrm{d}\ln e\,. \tag{75}$$

Introducing Equation (69):

$$\mathrm{d}T_\mathrm{d} \cong \frac{R_\mathrm{v} T_\mathrm{d}^2}{\varkappa l_\mathrm{v}}\,\frac{\mathrm{d}T}{T} = \frac{c_\mathrm{p} T_\mathrm{d}^2}{\varepsilon l_\mathrm{v}}\,\frac{\mathrm{d}T}{T}\,. \tag{76}$$

Writing $T_\mathrm{d} \sim T \sim 273\,\mathrm{K}$, and using finite differences, we obtain the approximate relation

$$\Delta T_\mathrm{d} \cong \tfrac{1}{6}\Delta T\,. \tag{77}$$

That is, T_d decreases approximately one sixth of the temperature drop, along an adiabatic ascent. Figure VII-13 shows this result on the vapor pressure diagram. The same relations are shown in Figure VII-14 on a tephigram. In this last representation, if we consider that r remains constant during the unsaturated ascent, it is obvious that T_d will slide along an equisaturated line (r = const.).

Let us now consider the variations of T and T_d during ascent from any level z_0 to the saturation level $z_\mathrm{s} = z_0 + \Delta z$. At the saturation level, T becomes equal to T_d and to T_s. Taking into account the lapse rate of T along an adiabat, and the derived relation for ΔT_d, we may write:

$$T - T_\mathrm{s} \cong 10\,\Delta z \quad (\Delta z \text{ in km})$$

$$T_\mathrm{d} - T_\mathrm{s} \cong \tfrac{1}{6}(10\,\Delta z)\,.$$

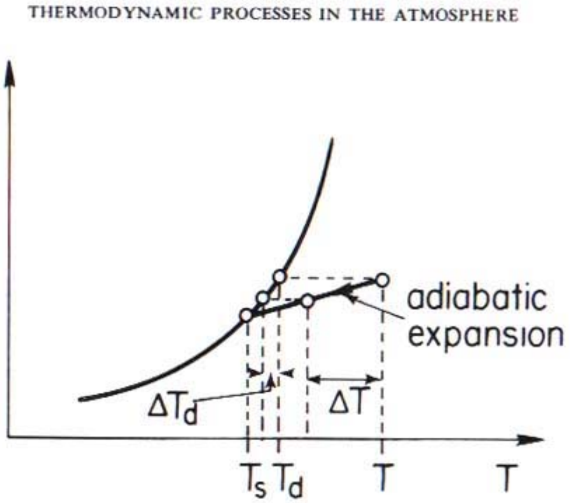

Fig. VII-13. Variation of dew-point and temperature during an adiabatic expansion, on a vapor pressure-temperature diagram.

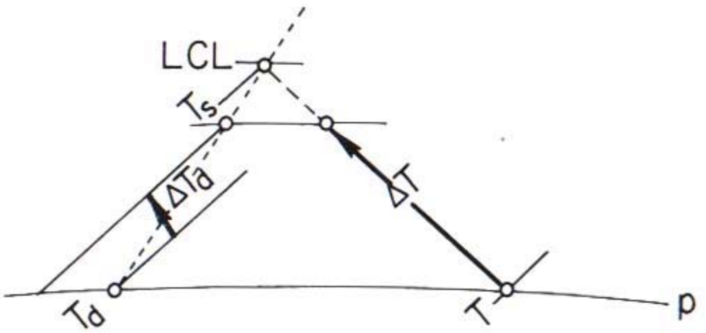

Fig. VII-14. Variation of dew-point and temperature during an adiabatic expansion, on a tephigram.

Subtracting:

$$T - T_d \cong \tfrac{50}{6}\,\Delta z$$

$$\Delta z \cong 0.12(T - T_d)\,\mathrm{km} \cong 400(T - T_d)\,\mathrm{ft}. \qquad (78)$$

This relation allows an approximate estimate of the height of the condensation level for an adiabatic ascent. This will give the base of cumuli, provided that these are actually formed by air rising from z_0.

The level at which an air mass attains saturation by adiabatic ascent is called the *lifting condensation level* (usually abridged LCL), and the corresponding temperature is the saturation temperature T_s.

7.8. Reversible Saturated Adiabatic Process

Our system will be a parcel of cloud which rises, expanding adiabatically and reversibly; it will be a closed system, keeping all the condensed water. Being adiabatic and reversible, the process is isentropic.

The same derivation and final formula will also be valid for an ice cloud, with the appropriate substitutions (sublimation latent heat for vaporization latent heat, specific heat capacity of ice for specific heat capacity of water, etc.).

We have already calculated the entropy expression for this system (Chapter IV, Equation (119)). It is:

$$S = (m_d c_{p_d} + m_t c_w) \ln T - m_d R_d \ln p_d + \frac{m_v l_v(T)}{T} + \text{const.} \tag{79}$$

Dividing by m_d and considering that the entropy must remain constant:

$$(c_{p_d} + r_{t,w} c_w) \ln T - R_d \ln p_d + \frac{r_w l_v}{T} = \text{const.} \tag{80}$$

where $r_{t,w} = m_t/m_d = r_w + m_w/m_d$, $p_d = p - e_w$; the subscript w indicates saturation values.

In differential form, Equation (80) becomes:

$$(c_{p_d} + r_{t,w} c_w)\, \mathrm{d} \ln T - R_d\, \mathrm{d} \ln p_d + \mathrm{d}\left(\frac{r_w l_v}{T}\right) = 0\,. \tag{81}$$

Equation (80), or (81), describes the reversible saturated adiabatic process. As we have seen before, our system is bivariant. If T and p_d are considered as the independent variables, r_w can be expressed (Chapter IV, Equation (79)) as a function of e_w: $r_w = \varepsilon e_w / p_d$; and e_w, in its turn, as a function of T, through Clausius-Clapeyron's integrated equation (Chapter IV, Section 8). Equation (80) gives thus a relation between the two independent variables (whichever we choose) and determines a curve in the plane T, p_d or in any usual diagram.

If in Equation (81) we neglect $r_{t,w} c_w$ as compared with c_{p_d} and e_w as compared with p_d, we obtain the approximate formula*

$$c_{p_d} \ln T - R_d \ln p + \frac{r_w l_v}{T} = \text{const.} \tag{82}$$

and considering that l_v varies slowly with T, we obtain the approximate differential formula

$$c_{p_d}\, \mathrm{d} \ln T - R_d\, \mathrm{d} \ln p + l_v\, \mathrm{d}\left(\frac{r_w}{T}\right) = 0\,. \tag{83}$$

7.9. Pseudoadiabatic Process

Equations (80) and (81) of Section 8 depend upon the value of r_t. This is a different constant for each different system considered, varying with the concentration of liquid

---

* For high temperatures, this becomes a rough approximation. If $r_{t,w}$ reaches a value of 0.05, $r_{t,w} c_w$ becomes equal to about 20% of c_{p_d}, and it can no longer be neglected.

water. For a given point T, p in a diagram, the vapor saturation mixing ratio r_w is also determined, but the liquid water mixing ratio m_w/m_d is an arbitrary parameter of the system. If we are considering rising air (expanding adiabatically), its value will depend on the level at which the air first became saturated. Therefore, through each point of a diagram an infinite number of reversible saturated adiabats will pass, differing (very little) from each other by the term depending on the liquid water content.

In order to avoid this inconvenience, aerological diagrams have recourse to another process of saturated expansion which gives uniquely defined curves. This is called the *pseudoadiabatic process*, and it is assumed in it that all the condensed water (or ice) falls out of the system as soon as it is produced. The system is an open one. However, we may easily find the relation in which we are interested, starting from the formulas of Section 8.

We must have at any instant

$$m_w = 0; \qquad r_{t,w} = r_w .$$

We may consider the condensation and elimination of a mass dm_w of water as occurring in two stages: (1) reversible saturated adiabatic expansion, with condensation of a mass dm_w of water, and (2) the water leaving the system. In the first stage we may apply Equation (81), substituting r_w for $r_{t,w}$:

$$(c_{p_d} + r_w c_w)\, d \ln T - R_d\, d \ln p_d + d\left(\frac{r_w l_v}{T}\right) = 0 . \tag{84}$$

In the second stage, the entropy of the system will decrease by $-s_w\, dm_w$, but this process will not affect the values of T and p. As for $r_{t,w}$, it will decrease by $-dm_w/m_d$, so that it will remain equal to $r_w(T)$. Therefore, Equation (84) describes the variation of T and p in a pseudoadiabatic process. If we want the integrated form, we shall have to remember that r_w depends now on T (which $r_{t,w}$ did not):

$$c_{p_d} \ln T + c_w \int r_w\, d \ln T - R_d \ln p_d + \frac{r_w l_v}{T} = \text{const.} \tag{85}$$

Obviously, in the reverse process (pseudoadiabatic saturated compression), one has to assume that water (or ice) is being introduced from outside at the instantaneous temperature of the system and at the necessary rate to maintain it exactly saturated.

If we now make the same approximations as in Section 8, we obtain again for pseudoadiabatic processes Equation (83), which is usually applied when precision is not required. Therefore, in approximate calculations no difference is made between the two processes. In aerological diagrams, the saturated adiabats correspond as explained to the pseudoadiabatic process. The cooling is slightly greater in pseudoadiabatic than in reversible expansion (for the same pressure change).

water. For a given point T, p in a diagram, the vapor saturation mixing ratio r_w is also determined, but the liquid water mixing ratio m_w/m_d is an arbitrary parameter of the system. If we are considering rising air (expanding adiabatically), its value will depend on the level at which the air first became saturated. Therefore, through each point of a diagram an infinite number of reversible saturated adiabats will pass, differing (very little) from each other by the term depending on the liquid water content.

In order to avoid this inconvenience, aerological diagrams have recourse to another process of saturated expansion which gives uniquely defined curves. This is called the *pseudoadiabatic process*, and it is assumed in it that all the condensed water (or ice) falls out of the system as soon as it is produced. The system is an open one. However, we may easily find the relation in which we are interested, starting from the formulas of Section 8.

We must have at any instant

$$m_w = 0; \qquad r_{t,w} = r_w .$$

We may consider the condensation and elimination of a mass $\mathrm{d}m_w$ of water as occurring in two stages: (1) reversible saturated adiabatic expansion, with condensation of a mass $\mathrm{d}m_w$ of water, and (2) the water leaving the system. In the first stage we may apply Equation (81), substituting r_w for $r_{t,w}$:

$$(c_{p_d} + r_w c_w)\,\mathrm{d}\ln T - R_d\,\mathrm{d}\ln p_d + \mathrm{d}\left(\frac{r_w l_v}{T}\right) = 0 . \tag{84}$$

In the second stage, the entropy of the system will decrease by $-s_w\,\mathrm{d}m_w$, but this process will not affect the values of T and p. As for $r_{t,w}$, it will decrease by $-\mathrm{d}m_w/m_d$, so that it will remain equal to $r_w(T)$. Therefore, Equation (84) describes the variation of T and p in a pseudoadiabatic process. If we want the integrated form, we shall have to remember that r_w depends now on T (which $r_{t,w}$ did not):

$$c_{p_d}\ln T + c_w \int r_w\,\mathrm{d}\ln T - R_d \ln p_d + \frac{r_w l_v}{T} = \text{const.} \tag{85}$$

Obviously, in the reverse process (pseudoadiabatic saturated compression), one has to assume that water (or ice) is being introduced from outside at the instantaneous temperature of the system and at the necessary rate to maintain it exactly saturated.

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7.10. Effect of Freezing in a Cloud

We shall consider now the effect of freezing, assuming that this occurs at a given level during the air ascent, i.e., at constant pressure. We shall also assume that we are dealing with an adiabatic, and therefore isenthalpic process. We can still call this the 'hail stage', but the initial temperature can have any value between 0 °C and −40 °C; it is an irreversible process.

As the water freezes, latent heat is released. Also, the air initially saturated with respect to water at the initial temperature T, will be supersaturated with respect to the frozen droplets; sublimation will occur, releasing an additional amount of latent heat, until the saturation vapor pressure with respect to ice is reached, at the final temperature T'.

As the enthalpy change only depends on the initial and final states, we can calculate it as if the total process occurred in three steps:

(1) Water freezes at constant temperature T.

(2) Vapor condenses on the ice at constant temperature T, until the water vapor pressure reaches the saturation value over ice at T'.

(3) The whole system is heated from T to T'.*

This can be schematically summarized as follows:

$$\begin{array}{l}\text{condensed phase:}\\ \text{temperature:}\\ \text{water vapor pressure:}\end{array}\left\{\begin{array}{l}\text{water}\\ T\\ e_{\mathrm{w}}(T)\end{array}\right. \xrightarrow{(1)} \left\{\begin{array}{l}\text{ice}\\ T\\ e_{\mathrm{w}}(T)\end{array}\right. \xrightarrow{(2)} \left\{\begin{array}{l}\text{ice}\\ T\\ e_{\mathrm{i}}(T')\end{array}\right. \xrightarrow{(3)} \left\{\begin{array}{l}\text{ice}\\ T'\\ e_{\mathrm{i}}(T')\end{array}\right.$$

The sum of the enthalpy changes for the three steps is the total change, and this must be zero.

We may consider for convenience a system containing unit mass of dry air, an amount r_{L} of liquid water ($r_{\mathrm{L}} = m_{\mathrm{w}}/m_{\mathrm{d}}$) and the saturation mixing ratio of water vapor at the initial temperature $r_{\mathrm{w}}(T)$. The three changes in enthalpy will be:

$$\Delta H_1 = -\,l_{\mathrm{f}} r_{\mathrm{L}} \tag{86}$$

$$\Delta H_2 = -\,l_{\mathrm{s}}[r_{\mathrm{w}}(T) - r_{\mathrm{i}}(T')] \tag{87}$$

$$\begin{aligned}\Delta H_3 &= [c_{p_{\mathrm{d}}} + r_{\mathrm{i}}(T')c_{p_{\mathrm{v}}} + r_{\mathrm{S}}c_{\mathrm{i}}]\,(T' - T)\\ &= c_{\mathrm{p}}(T' - T)\end{aligned} \tag{88}$$

where $r_{\mathrm{i}}(T')$ is the saturation mixing ratio over ice at the final temperature T', $r_{\mathrm{S}} = r_{\mathrm{L}} + r_{\mathrm{w}}(T) - r_{\mathrm{i}}(T')$ is the final mass of ice, and we have called c_{p} the average specific heat capacity of the final system.

---

* It can be imagined that the latent heat released in (1) is given to an external source, and that in (3) the necessary heat is again received from an external source.

We can express the mixing ratios in Equation (87) in terms of vapor pressures (formula in Chapter IV, Equation (82)):

$$r_w(T) \cong \frac{\varepsilon e_w(T)}{p} \tag{89}$$

$$r_i(T') \cong \frac{\varepsilon e_i(T')}{p} \tag{90}$$

and use the Clausius-Clapeyron equation to express e_i as a function of the initial temperature, assuming that $(T'-T)$ is small enough to be treated as a differential:

$$e_i(T') = e_i(T) + \frac{l_s e_i(T)}{R_v T^2}(T' - T). \tag{91}$$

Introducing Equations (89), (90), (91) into (87), we have

$$\begin{aligned} \Delta H_2 &= -\frac{\varepsilon l_s}{p}\left[e_w(T) - e_i(T) - \frac{l_s e_i(T)}{R_v T^2}(T' - T)\right] \\ &= -r_w(T) l_s\left[1 - \frac{e_i(T)}{e_w(T)}\right] + \frac{r_i(T) l_s^2}{R_v T^2}(T' - T). \end{aligned} \tag{92}$$

Introducing Equations (86), (88) and (92) in

$$\Delta H_1 + \Delta H_2 + \Delta H_3 = 0 \tag{93}$$

and solving for $(T'-T) = \Delta T$, we finally obtain

$$\Delta T = \frac{l_f r_L + l_s r_w\left(1 - \frac{e_i}{e_w}\right)}{c_p + \frac{r_i l_s^2}{R_v T^2}}. \tag{94}$$

Here the saturation values are taken at T. The term c_p contains $r_i(T')$, which is not known if we are calculating ΔT; however $r_i(T')c_{p_v} \ll c_{p_d}$ (cf. Equation (88)) and $r_i(T)$ can be used for $r_i(T')$. If needed, successive approximations would rapidly improve the computation.

Formula (94) gives the increase in temperature due to the freezing of the cloud water. For individual cloud parcels, this increase adds to the buoyancy (upward thrust due to the difference of density with the environment: cf., Chapter IX, Section 2) which is important in the dynamics of such a cloud. In reality, however, freezing does not occur suddenly at one level, but gradually over a temperature interval. If this were to be taken into account, the problem should be integrated with that of the saturated expansion. The distribution of freezing along the ascent should then be known or assumed. The system would not be in equilibrium and the vapor pressure would have some intermediate value between those of saturation over water and saturation over ice. The solution of the problem (curve $T = f(p)$) would depend on the model by which the process might be approximated.

Especially at low temperatures, the warming accompanying freezing will depend critically on the liquid water content of the cloud. In the limiting case, when r_w and r_i become negligible,

$$\Delta T = \frac{r_L}{c_p} l_f. \tag{95}$$

At $-30\,°\mathrm{C}$, if $r_L = 10\ \mathrm{g\ kg^{-1}}$ (a high liquid water content), Equation (95) gives $\Delta T = 3.3\ \mathrm{K}$. This warming could be a very significant factor in cloud development.

7.11. Polytropic Expansion

Vertical motions in the atmosphere can have velocities varying within a very wide range. Violent convective processes may entail updrafts of several tens of meters per second. On the other extreme, some synoptic situations are associated with very slow motions of horizontally extended layers; vertical velocities can be as small as 0.1 to 1 cm $\mathrm{s^{-1}}$. In the latter extreme cases the expansion or compression cannot be considered as strictly adiabatic, as it has been assumed so far for vertical displacements; the exchange of radiant energy in 'long wave' (i.e. in the terrestrial and atmospheric range of wave lengths, from 4 to several tens of micrometers) with other horizontal layers, with the ground and with outer space cannot be ignored any more without introducing appreciable error. Such radiant exchanges produce temperature changes in the air of the order of 1–2 K $\mathrm{day^{-1}}$. For the sake of comparison, the adiabatic cooling of air rising at $1\ \mathrm{cm\,s^{-1}} \cong 0.86\ \mathrm{km\,day^{-1}}$ would be $8.4\ \mathrm{K\,day^{-1}}$ (as it will be seen in Chapter IX, Section 3) and that of air rising at $0.1\ \mathrm{cm\,s^{-1}} = 86\ \mathrm{m\,day^{-1}}$ would be $0.84\ \mathrm{K\,day^{-1}}$.

These non-adiabatic expansions or compressions can be approximated by the polytropic processes described in Chapter II, Section 8. In order to find the relation between the values of the coefficients k and n in the formulas (61) of that section and the conditions of these atmospheric processes, we can write the first principle in specific quantities as

$$\delta q = c_p\,\mathrm{d}T - v\,\mathrm{d}p, \tag{96}$$

differentiate

$$T = \text{const.}\ p^{(1-1/n)} = \text{const.}\ p^{k} \tag{97}$$

logarithmically, which gives

$$\mathrm{d}T = k\frac{T}{p}\,\mathrm{d}p \tag{98}$$

and make use of the hydrostatic equation

$$\mathrm{d}p = -g\varrho\,\mathrm{d}z \tag{99}$$

(where g = gravity, dz = air ascent), which will be studied in Chapter IX, Section 2. By eliminating dT and dp from the three equations (96), (98) and (99), it is easily found that

$$k = \varkappa\left(1 - \frac{1}{g}\frac{\delta g}{dz}\right) \tag{100}$$

and

$$n = \frac{1}{1-k} = \frac{1}{1 - \varkappa\left(1 - \frac{1}{g}\frac{\delta g}{dz}\right)}. \tag{101}$$

$\delta q/dz$ is the heat absorbed by the air per unit mass and unit length of ascent. If the information available is given in terms of rate of temperature charge λ due solely to radiation exchange

$$\lambda = \frac{dT}{dt} = \frac{1}{c_p}\frac{\delta q}{dt} \tag{102}$$

and velocity of ascent

$$U = \frac{dz}{dt} \tag{103}$$

we can write

$$\frac{\delta q}{dz} = \frac{c_p \lambda}{U}. \tag{104}$$

Thus, coming back to the figures mentioned before, if for instance $\lambda = \pm 2\,\mathrm{K\,day^{-1}}$ (upper and lower signs corresponding to warming and cooling, respectively) and $U = 0.86\,\mathrm{km\,day^{-1}}$, we obtain (assuming dry air), with $c_p = 1005\ \mathrm{J\,kg^{-1}\,K^{-1}}$, $g = 9.81\,m\,s^{-2}$:

$$\frac{\delta q}{dz} = \pm 2.34\,\mathrm{J\,kg^{-1}\,m^{-1}}$$

and $k = 0.219$ or 0.355, corresponding to the upper or the lower sign, respectively; this is to be compared with the adiabatic coefficient $\varkappa_d = 0.286$. The corresponding values of n will be 1.28 and 1.55, respectively, to be compared with $\eta_d = 1.40$.

7.12. Vertical Mixing

We shall consider now the mixing of air masses along the vertical (by turbulent and/or convective processes). The analysis becomes complicated in this case by the continuous variation of p, T and r with height. We shall consider first two isolated masses m_1 and m_2 at the pressure levels p_1 and p_2 (with temperatures T_1 and T_2), which move to another pressure p and mix. We have then a first stage of adiabatic expansion or compression, for which

$$T_1' = T_1\left(\frac{p}{p_1}\right)^{\varkappa}; \qquad T_2' = T_2\left(\frac{p}{p_2}\right)^{\varkappa}. \tag{105}$$

The specific humidities q_1 and q_2 (or the mixing ratios r_1 and r_2) are preserved in this stage.

The second stage will be an isobaric adiabatic mixture, already considered in Section 4; formulae (35) (or the approximate Equation (37)), (36) and (39) will apply, as well as formulae similar to those of Equation (35) and (37) for the potential temperature; in particular Equation (38), where θ_1 and θ_2 are the initial values, which do not vary during the adiabatic expansion or compression. That is, the potential temperature of the mixture is approximately given by the weighted average of the potential temperatures of the two air masses. The third stage will consist in taking the two air masses to their original pressure levels. The location of the auxiliary level p used for the derivation is of course immaterial.*

Let us consider now an atmospheric layer with a thickness $\Delta p = p_2 - p_1$, and let us suppose that it mixes vertically. We can imagine that the process is performed by bringing the whole layer to the same level p, mixing it isobarically and redistributing it in the original interval Δp. The mixture will consist of air with a potential temperature θ equal to the weighted average for the whole layer. The vertical redistribution will preserve the value of θ, since it consists of adiabatic expansions or compressions. Therefore, when the layer is thoroughly mixed, θ will be constant with height.

The mass per unit area of an infinitesimal layer $\mathrm{d}z$ will be $\mathrm{d}m = \varrho\,\mathrm{d}z$, where ϱ is the density at the height of the layer. We shall see in Chapter VIII, Section 2 that (assuming hydrostatic equilibrium)

$$\varrho\,\mathrm{d}z = -\frac{1}{g}\,\mathrm{d}p\,, \tag{106}$$

where g is the acceleration of gravity and $-\mathrm{d}p$ the variation of p in $\mathrm{d}z$. Therefore

$$\bar{\theta} = \frac{\int_0^m \theta\,\mathrm{d}m}{m} = \frac{\int_0^z \theta\varrho\,\mathrm{d}z}{\int_0^z \varrho\,\mathrm{d}z} = -\frac{\int_{p_1}^{p_2} \theta\,\mathrm{d}p}{p_1 - p_2}\,. \tag{107}$$

Similar expressions would give the final values of q and r. The temperature T will have a distribution along p given by the adiabat for $\bar{\theta}$, i.e.,

$$T = \bar{\theta}\left(\frac{p}{1000}\right)^{\varkappa}. \tag{108}$$

---

* We might also have assumed that both masses move to $p = 1000$ mb, in which case $T_1' = \theta_1$ and $T_2' = \theta_2$.

This case is illustrated by Figure VII-15 on a tephigram, where the solid line shows the initial temperature distribution, the dashed line is the final adiabat $\bar{\theta}$,* and the dotted line (vapor line corresponding to the final uniform value of $\bar{r}$) shows that the final distribution does not reach the saturation level.

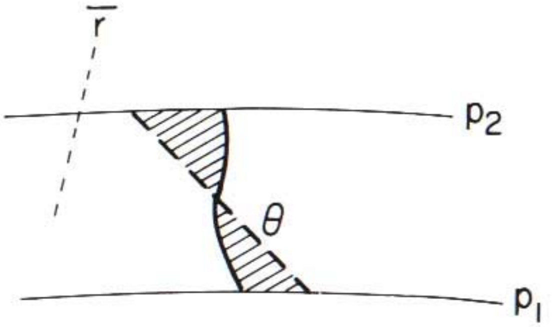

Fig. VII-15. Vertical mixing without condensation, on a tephigram.

If the vapor line $\bar{r}$ intersects the adiabat $\bar{\theta}$, from that level upwards condensation will occur and the final temperature distribution will follow the saturated adiabat. The intersection level is called the *mixing condensation level* (MCL). This case is illustrated by Figure VII-16.

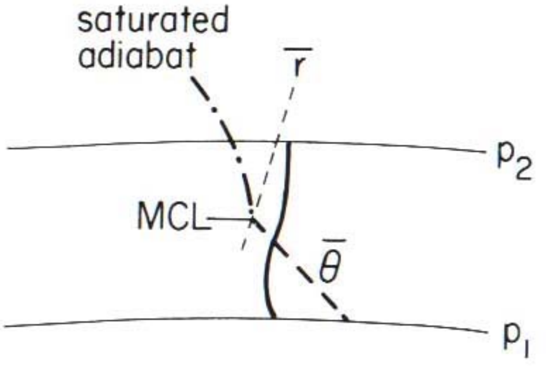

Fig. VII-16. Mixing condensation level, on a tephigram.

7.13. Pseudo- or Adiabatic Equivalent and Wet-Bulb Temperatures

Formula (29) is not convenient for a determination of T_{ie} and T_{iw} on aerological diagrams. But another two parameters, closely related to the above, may be defined, that can be easily found on the diagrams.

Let P be the image point of the air, with a mixing ratio r (see Figure VII-17). Let us

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* Compensation of areas is not strictly applicable here, but it will give a good approximation if the thickness of the layer is not too large.

assume that the air expands adiabatically until it reaches saturation; we have seen in Section 7 that this is always possible. This will happen at the intersection of the dry adiabat with the vapor line r. This is the *characteristic point* of the air, and has been designated as P_c on the diagram. Its temperature is the saturation temperature T_s. A saturated adiabat must also pass through that point; if we now follow it towards increasing values of the pressure, we attain the point P_w at the original pressure p. The temperature of P_w is called the *adiabatic wet-bulb temperature* or the *pseudo-wet-bulb temperature*. In order to be able to follow the curve P_cP_w, we must imagine that we are evaporating water into the air, so as to keep it saturated at increasing temperatures. According to the definition of these curves on the diagrams (pseudoadiabatic process, Section 9), we must consider that liquid water is being introduced in the precisely necessary amounts to maintain saturation, at the varying temperature of the air, and evaporated into it. If we compare this process with the experimental definition of T_{iw} (through the wet-bulb psychrometer) we find that while in the latter case all the added water was initially at the temperature T_{iw} (and its heat capacity was finally neglected in the analytical definition of T_{iw}), in the process now under consideration it was initially at varying temperatures between T_s and T_{aw} and after evaporation it had to be warmed to T_{aw}, subtracting heat from the air. For this reason T_{aw} is smaller than T_{iw}, but only by a small difference, usually not exceeding 0.5 °C.

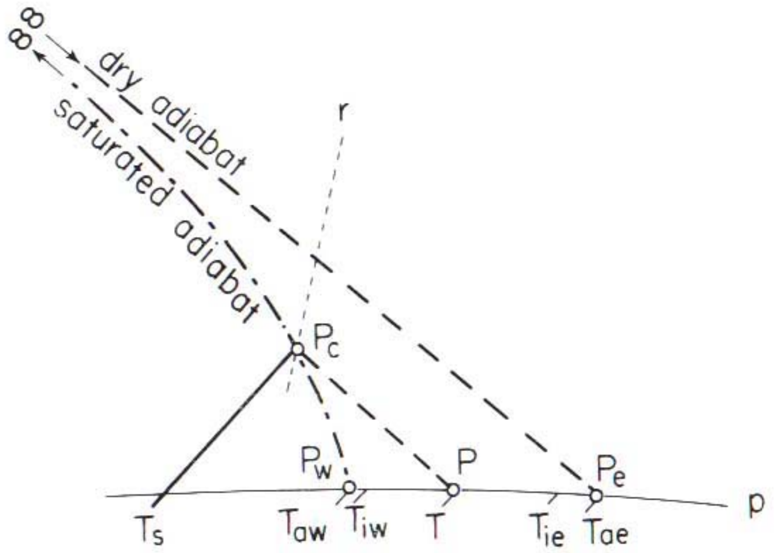

Fig. VII-17. Wet-bulb and equivalent temperatures.

Let us assume now that again we expand the air, starting from P. But this time we continue expanding after P_c. The air will then follow the saturated adiabat, the water vapor will condense, and we assume that it falls out of the system during the process. If we continue this process indefinitely, the curve will tend asymptotically to a dry adiabat, as the water vapor content becomes negligible (this is indicated in Figure VII-17 by ∞). Along this stage, water vapor cools together with the air before condensing and falling out at the variable temperature of condensation. Once all the vapor has thus been eliminated, we compress again the air to the original pressure p. In this last stage, the air is dry and will follow a dry adiabat, reaching the final temperature T_{ae} which we call the *adiabatic equivalent* or *pseudo-equivalent temperature*. As for T_{ie}, the change (not the process) undergone by the dry air component of the system is isobaric. But while for T_{ie} the condensed water remains at intermediate temperatures between T and T_{ie} (and its heat capacity is finally disregarded), in the process now under consideration it remains at varying temperatures below T_s; thus, $T_{ae} > T_{ie}$. The difference is in this case larger than for the wet-bulb temperatures, and cannot be neglected in general.

If in the case of the pseudo-wet-bulb temperature we extend the saturated adiabat to the 1000 mb isobar, the intersection determines, by definition, the *pseudo-wet-bulb potential temperature* θ_{aw}. Similarly, by extending the dry adiabat from T_{ae} to the intersection with the 1000 mb isobar, we determine the *pseudo-equivalent potential temperature* θ_{ae}.

7.14. Summary of Temperature and Humidity Parameters. Conservative Properties

We summarize now the different temperatures that have been defined, with reference to the sections where they were introduced.

T = temperature

Dry temperatures

T_v = virtual temperature (Chapter IV, Section 11).

$T_e = T_{ie}$ = (isobaric) equivalent temperature (Section 3).

T_{ae} = adiabatic equivalent, or pseudo-equivalent, temperature (Section 13).

Saturation temperatures

T_d = dew point temperature (Section 1).

T_f = frost point temperature (Section 1).

$T_w = T_{iw}$ = (isobaric) wet-bulb temperature (Section 3).

T_{aw} = adiabatic wet-bulb, or pseudo-wet-bulb, temperature (Section 13).

T_s = saturation temperature (Section 7).

Potential temperatures

θ = potential temperature (Chapter II, Section 7).

θ_v = virtual potential temperature (Chapter IV, Section 13).

$\theta_e = \theta_{ie}$ = (isobaric) equivalent potential temperature.*

θ_{ae} = adiabatic equivalent, or pseudo-equivalent, potential temperature (Section 13).

$\theta_w = \theta_{iw}$ = (isobaric) wet-bulb potential temperature.*

θ_{aw} = adiabatic wet-bulb, or pseudo-wet-bulb, potential temperature (Section 13).

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* Defined in a similar way to the corresponding pseudo-potential temperature (see Figure VII-18).

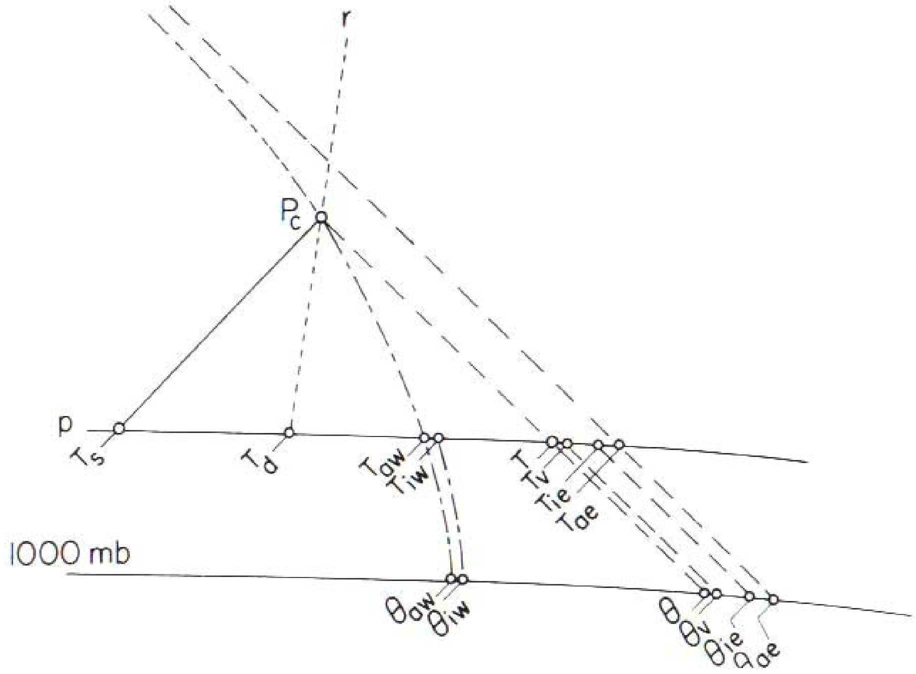

Fig. VII-18. Summary of temperature-humidity parameters.

Among the non-potential temperatures we have the following inequality relations:

$$T_s < T_d < T_{aw} < T_{iw} < T < T_v < T_{ie} < T_{ae}.$$

Figure VII-18 gives a summary of these parameters, as they are determined on a tephigram.

The reason why all these parameters are defined lies in their conservative properties regarding certain meteorological processes. Thus, for instance, an air mass becoming warmer or cooler, isobarically, maintains the same value of its dew point. An air mass rising or descending adiabatically, without condensation or evaporation, keeps constant its potential temperature; if evaporation or condensation takes place, θ varies but θ_{aw} and θ_{ae} preserve their values. A cool, moist, air mass that rises along the slopes of a mountain range, drying by precipitation, may eventually descend along the slopes on the lee side, arriving as a dry, hot air mass (Föhn); the air mass may be recognized to be the same as before if its potential temperatures θ_{aw} or θ_{ae} remain the same.

These parameters may thus be used to identify air masses undergoing a series of transformations.

Table VII-1 summarizes the conservative properties of several temperature and humidity parameters, with respect to the main processes referred to above.

TABLE VII-1

Conservative properties of several parameters, C = conservative; NC = non-conservative

Property	Process:			
	Isobaric warming or cooling (without condensation or vaporization)	Isobaric vaporization or condensation	Non-saturated adiabatic expansion	Saturated adiabatic expansion
U_w	NC	NC	NC	C
e or T_d	C	NC	NC	NC
q or r	C	NC	C	NC
T_{aw} or T_{ae}	NC	C	NC	NC
θ	NC	NC	C	NC
θ_{aw} or θ_{ae}	NC	C	C	C

PROBLEMS

1. An air mass has a temperature of 20 °C at 1000 mb pressure, with a mixing ratio of 10 g kg^{-1}. After a certain time, the same mass, without mixing with the environment has acquired a temperature of 10 °C and a pressure of 750 mb. Knowing that the dew-point is initially 14.0 °C, calculate analytically the initial and final values of the vapor pressure e and relative humidity U_w, and the final dew-point temperature T_{d_f}. Do not use tables of saturated vapor pressures.
2. During the formation of a radiation fog, 1 cal g^{-1} is lost after saturation started, at 10 °C. The pressure is 1000 mb. What is the final temperature? What was the decrease in vapor pressure, and what is the concentration of the fog, in grams of liquid water per cubic meter?

   Make an approximate calculation, treating the differences as differentials, and using the approximate value 12 mb for the saturation vapor pressure.
3. The isobaric equivalent temperature is defined as the temperature that the humid air would attain if all its water vapor were condensed out at constant pressure, and the latent heat released used to warm the air. Using the expression for the enthalpy of Chapter IV, Equation (104), show that a more accurate expression of the isobaric equivalent temperature than Equation (27) is given by the following formula

$$\ln \frac{l_v(T_{ie})}{l_v(T)} = \frac{c_{p_v} - c_w}{c_{p_v}} \ln \left(1 + \frac{c_{p_v}}{c_{pd}} r\right)$$

4. An air parcel, initially at 10 °C and with a water vapor pressure of 3.5 mb, undergoes an isobaric, adiabatic wet-bulb process to saturation. Derive in approximate form (assuming the heat capacity of the water substance is negligible) an expression giving the slope of the line representing this process on a $e-T$ (vapor pressure – temperature) diagram. Draw this line on such a diagram for a total pressure of 1000 mb, and obtain the approximate values of the wet-bulb temperature and of the equivalent temperature.

5. Two equal masses of air, both at 1000 mb, mix thoroughly. Their initial temperatures and mixing ratios are: $T_1 = 23.8\,°\text{C}$, $r_1 = 16.3\ \text{g kg}^{-1}$, $T_2 = -6.4\,°\text{C}$, $r_2 = 1.3\ \text{g kg}^{-1}$. Describe the final result, expressing: the temperature, the mixing ratio, if air is saturated or not and, if so, the liquid water content in g m^{-3}. Make all calculations analytically; if necessary, use a table of saturated vapor pressures or of saturation mixing ratios.
6. Two equal masses of air, one at 0 °C and the other at 25 °C, both saturated but without any liquid water, mix thoroughly. The pressure is 1000 mb. Calculate the liquid water content of the resulting fog, in g m^{-3}. Use tables as needed, but no diagrams.
7. On a given occasion, convective activity becomes visualized by isolated cumuli whose bases are at 2000 m above the ground. Near the ground, the temperature and dew point have been constant for the last hour, with values 30 °C and 13.3 °C, respectively. Would you say that the cumuli might have been formed by air rising from near the ground? Explain.
8. Complete the tephigram of Problem VI-1 by drawing three saturated adiabats: $\theta_{\text{aw}} = 250$, 270 and 290 K.
9. A parcel of cloud air rises, expanding adiabatically, from 1000 mb, 20 °C, until it reaches a temperature of -40 °C. What is the final pressure? Make the following approximations:
   (a) Neglect the remaining mixing ratio of water vapor at -40 °C,
   (b) Neglect the heat capacity of water (both liquid and vapor),
   (c) Neglect the partial pressure of water vapor against the total pressure, and
   (d) Assume a constant value of the latent heat of vaporization (use an average value between -40 °C and $+20$ °C).

   Use tables as needed. Do not use a tephigram. The calculation must be made analytically.
10. Consider a mass of air which is aloft above the tropics, saturated with water vapor with respect to ice, at 230 mb and -40 °C. Assume that it rises adiabatically until reaching the tropopause at 120 mb. Water vapor condenses to ice. What is the temperature (°C), within 1° of approximation, at the tropopause?

    The heat capacity of ice can be neglected; the water vapor pressure is negligible as compared with that of dry air; the water vapor at the tropopause can be neglected altogether (i.e., set $\cong 0$). You can use the Table of Constants in the book.
11. Saturated air rises adiabatically, water vapor condensing into water droplets while the air remains saturated. Make a simple sketch on a vapor pressure diagram (e, T) where you show the initial and final points, the path representing the process, and a second path to the same final point, consisting in
    (a) adiabatic expansion without condensation, followed by
    (b) condensation at constant pressure.
12. (a) Derive a formula for the (specific) work of expansion associated with a polytropic ascent of air, as a function of its temperature variation ΔT and the polytropic exponent n.

(b) Calculate that work, in J kg^{-1}, for $\Delta T = -10$ K and $n = 1.2$. How much is the corresponding absorbed heat, per kg?

13. Derive the expression for the final temperature distribution $T = f(p)$ acquired by an isothermal layer of temperature T_0 contained between the isobars p_1 and p_2, when it is thoroughly mixed vertically. Assume that there is no condensation.

14. With the data of Problem VI-4, find on a tephigram the adiabatic wet-bulb temperature T_{aw}, the potential adiabatic wet-bulb temperature θ_{aw} and the saturation temperature T_s and pressure p_s (temperature and pressure at which condensation starts, when expanded adiabatically). Give the temperatures in °C.

15. (a) Derive an expression for θ_{ae} as a function of T and p(r_w, e_w or T_s can be left as implicit functions of T, p). Use the usual approximate formula for saturated adiabats.

(b) What is the value of θ_{ae} for saturated air at 800 mb, 0 °C, as obtained with the derived expression?

16. With the data of Problem IV-7, determine graphically, on a tephigram, $r_w, \theta, \theta_v, p_s, T_s, T_{aw}, \theta_{aw}$ and T_d. Compute T_{iw} and T_{ie}.

17. A Föhn blowing on the ground at 1000 mb has a temperature of 38 °C and a mixing ratio of 4 g kg^{-1}. Could this air be the same as that at the 1000 mb level on the windward side of the mountains, with a temperature of 21.5° and a mixing ratio of 10 g kg^{-1}? Could it be the same as that at the 800 mb level with 5 °C and 5 g kg^{-1}? Use the tephigram, and give the reasons for your answer.

18. A mass of air undergoes vertical displacements, during which precipitation falls out. If it can be assumed that no appreciable heat has been exchanged with the environment, what invariant parameter could be used to identify the mass through its transformations? What invariant parameter could be useful to identify a non-precipitating air mass moving horizontally over land and changing its temperature?