On the Dynamic Interpretation of the Virtual Temperature

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ABSTRACT

The concept of virtual temperature is reviewed and extended into the definition of the dynamic virtual temperature, which is the temperature that a parcel of dry air should have in order to experience the same acceleration as a parcel of cloud air. It is obtained from the equations of motion and depends on the water content in the three thermodynamic states: vapor, liquid, and solid. The scale analysis of the equation of the dynamic virtual temperature shows that the terms due to the acceleration and phase transitions of the particles are negligible with respect to the terms depending on gravity. Therefore, even though conceptually more adequate, the approximate mathematical expression of the dynamic virtual temperature is practically identical to the conventional definition of virtual temperature accounting for water loading.

1. Introduction

The concept of virtual temperature has been introduced in meteorology to account for the dependence of air density on water vapor content and represents the temperature of dry air that has the same density as a parcel of moist air at the same pressure (Guldberg and Mohn 1876; Dufour 1963; cf. Curry and Webster 1999). The virtual temperature is then used in the computation of the buoyancy force.

Saunders (1957) generalized the concept of virtual temperature to cloudy air. He assumed that the observed decrease of buoyancy of cloudy air was related to the increase in air density due to the presence of hydrometeors. Therefore, Saunders' cloudy virtual temperature represents the temperature of dry air having the same density as a cloud composed of moist air and condensed water at the same pressure. Since then, this temperature has been largely used to calculate the convective available potential energy (CAPE) in several convective models as well as to establish the vertical stability in the atmosphere (Emanuel 1994; Curry and Webster 1999; Jacobson 2000).

In the definition of cloudy virtual temperature the acceleration of the air and the hydrometeors, the variation of momentum due to hydrometeor interactions and phase transitions have been ignored even though the influence of the condensed water on the vertical motion of moist air is rather a consequence of the momentum exchanges between moist air and water particles as the studies on cloud dynamic and precipitation demonstrate (Ogura 1963; Arnaldson et al. 1968; List and Lozowski 1970; Clark and List 1971). Recent textbooks are aware of the importance of hydrometeor drag, which is introduced in the vertical motion equation for moist air as a force proportional to the weight of water particles, the so-called liquid water loading (e.g., Cotton and Anthes 1989; Houze 1993; Emanuel 1994). Bannon (2002) considers the assumptions underlying the estimate of the hydrometeor drag force by their weight and proves the pertinence of this approximation. He obtains the dynamic and thermodynamic equations for cloudy air as a multiphase, multitemperature, and multivelocity system allowing transitions between water phases and assuming the dynamic equilibrium of the water particles falling at terminal velocity. He shows that the net microphysical momentum forcing of the moist air is mainly due to the drag force and

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to the momentum exchange associated with phase changes.

On the one hand, Bannon's vertical momentum equation for moist air includes all the important physical phenomena at play, but, on the other hand, the definition of virtual temperature, largely employed to calculate the buoyancy force, is based on a simplified and unphysical representation of the phenomenon suggesting the idea of a change in the bulk density of air due the presence of condensed water. The objective of this work is to clarify the physical meaning of virtual temperature defining it as the temperature of the dry air that has the same acceleration as the moist air surrounding the water particles at the same pressure. This temperature accounts for the influence of water (vapor, liquid, and solid) on the vertical acceleration of the air and is deduced from the vertical momentum equation for moist air.

Following Bannon (2002), in section 2, we present the momentum equation of the moist air as part of a multicomponent and multiphase flow. In section 3, we define the virtual temperature according to this equation. A scale analysis is then performed in section 4 to evaluate the order of magnitude of the terms dependent on the condensed water.

2. Basic equations and assumptions

a. Virtual temperature for cloudy air

According to Saunders (1957), a cloud parcel is seen as a heterogeneous closed system formed by dry air, water vapor, and condensed water. By definition, the virtual temperature for cloud air, $T_{v,c}$, is the temperature required for dry air to yield the same density as cloudy air and is given by (cf. Emanuel 1994)

$$T_{v,c} \cong T(1 + 0.608q_v - q_w), \tag{1}$$

where T is the air temperature, q_v the specific humidity, and q_w the condensed water mass per unit mass of moist air. This definition of $T_{v,c}$ is based on the premise that the hydrometeors are tied to the moist air and that buoyancy decreases because of the increase in the air density.

b. Momentum equation for the moist air

Bannon's (2002) Eq. (5.15) identifies the main terms linked to the contribution of water to the acceleration of moist air. The derivation of this equation assumes that 1) moist air is a mixture of dry air and water vapor, two ideal gases; 2) the diffusion velocity of water vapor relative to air is negligible; 3) the volume filled with the dispersed phases is small compared to the total volume of the system; and 4) the hydrometeors are in dynamic equilibrium. The generalization of this equation for any population of hydrometeors of different sizes and phases gives

$$(1 + r_v)\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho_a}\boldsymbol{\nabla}p + (1 + r_v)\mathbf{g} + \frac{\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}}{\rho_a} + \mathbf{g}\sum_k r_k - \left(\sum_k \mathbf{u}_k\frac{Dr_k}{Dt_k} + \mathbf{u}\frac{Dr_v}{Dt}\right), \tag{2}$$

where $\mathbf{u}$ is the moist air velocity, ρ_a the density of dry air, $\mathbf{g}$ the gravity acceleration, r_v the mixing ratio of water vapor, r_k the mixing rate of the hydrometeor k of density ρ_k, velocity $\mathbf{u}_k$, and equivalent diameter D_k. Here, $D(\cdot)/Dt = \partial(\cdot)/\partial t + \mathbf{u}\cdot\boldsymbol{\nabla}(\cdot)$ and $D(\cdot)/Dt_k = \partial(\cdot)/\partial t + \mathbf{u}_k\cdot\boldsymbol{\nabla}(\cdot)$ are the material derivatives with respect to the motion of air and of the particle k, respectively. The moist air momentum changes under the action of the pressure gradient force, the gravity force, the divergence of the viscous stress vector of moist air $\boldsymbol{\sigma}$, the drag force exerted by all the hydrometeors, and the momentum change associated with phase changes.

3. Dynamic virtual temperature

For a nonsheared flow ($\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = 0$), we obtain from (2) that the vertical component of the momentum equation of the moist air is

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g - g\sum_k q_k + B_z, \tag{3}$$

where $\rho = \rho_a(1 + r_v)$ is the density of moist air, w is the vertical component of its velocity, q_k is the mass of all the particles of order k per unit of mass of moist air q_v, and $B_z = -(\Sigma_k w_k Dq_k/Dt_k + w\,Dq_v/Dt)$ accounts for the effects of phase changes. The term $-g\,\Sigma_k q_k = -gq_w$ in the right-hand side of (3) represents the drag force of the particles on the moist air.

In a hydrostatic atmosphere, the gravity, and the pressure gradient force can be represented in term of the virtual temperature of the moist parcel, $T_v = T(1 + 0.608q_v)$, and of the surrounding air, $T'_v = T'(1 + 0.608q'_v)$ (cf. Curry and Webster 1999). Therefore, from (3) we obtain

$$\frac{Dw}{Dt} = \left(\frac{T_v}{T'_v} - 1\right)g - gq_w + B_z. \tag{4}$$

We now define the dynamic virtual temperature, $T_{v,d}$, as the temperature that dry air should have to experience the same vertical acceleration as a parcel of moist air in presence of condensed water. Therefore, it follows that the momentum equation for the parcel is

$$\frac{Dw}{Dt} = \left(\frac{T_{v,d}}{T'_v} - 1\right)g. \tag{5}$$

It should be stressed that T'_v in (5) is associated with the vertical pressure gradient, which does not depend on the condensed water load of the air surrounding the parcel. Some authors use the environment density temperature that includes the condensed phase to calculate the gradient pressure force (Emanuel 1994). Even though conceptually incorrect, there are no practical implications from this approximation because the difference between the virtual temperature of the environment and its density temperature is equal to $T' q'_w$, where the prime sign refers to the environment conditions, which is of the same order of magnitude that the uncertainty associated with the measurement of the temperature itself. Equating (4) and (5) and solving for $T_{v,d}$, we obtain

$$T_{v,d} = T'_v\left(\frac{T_v}{T'_v} - q_w - \frac{B_z}{g}\right) = T_v - T'_v q_w + T'_v\frac{B_z}{g}. \tag{6}$$

Because $O(q'_v) \approx O(q_v) \approx 10^{-2}$, $O(q_w) \leq O(q_v)$, and, as shown in section 4, $O(B_z/g) < O(q_v)$ in the development of (6) we ignored the terms containing the products $q'_v q_v$ and $q'_v B_z/g$. Then $T_{v,d}$ can be approximated by

$$T_{v,d} \cong T(1 + 0.608q_v - q_w + B_z/g). \tag{7}$$

In a flow where all the hydrometeors move at equilibrium velocity and there are no phase transitions $B_z = 0$ and from (1) and (7) we obtain that $T_{v,d} \approx T_{v,c}$.

4. The order of magnitude of the contribution of condensed water

From (4) the contribution of condensed water to the vertical acceleration of moist air is due to the total weight of the hydrometeors and phase transitions:

$$A_z = -gq_w + B_z = \underbrace{-gq_w}_{\text{Gravity}} \underbrace{- \sum_k w_k \frac{Dq_k}{Dt_k} - w\frac{Dq_v}{Dt}}_{\text{Phase transitions}}. \tag{8}$$

We consider next the order of magnitude of the different terms in (8).

a. The gravity contribution

The order of magnitude of the condensed water content, q_w in a cloud is $\sim 10^{-3}$. Therefore, the order of magnitude of the gravity term is

$$O(-gq_w) \sim 10^{-2}\ (\text{m s}^{-2}). \tag{9}$$

b. Phase transitions

There are two situations to consider: 1) the growth by condensation of the population of hydrometeors, and 2) the evaporation of the precipitation in a layer of unsaturated air.

1) GROWTH BY CONDENSATION

To proceed to the scale analysis we assume a cloud characterized by a population of n (m^{-3}) hydrometeors of same size and mass, moving at equilibrium velocity, $w_k = w + w_T$, where w is the vertical velocity of the air parcel and w_T is the terminal velocity of the hydrometeor. In this case the acceleration due to condensation is

$$-\sum_k w_k \frac{Dq_k}{Dt_k} - w\frac{Dq_v}{Dt} \cong -(w + w_T)\frac{Dr_w}{Dt_p} - w\frac{Dr_v}{Dt}, \tag{10}$$

where $D(\cdot)/Dt_p = \partial(\cdot)/\partial t + (w + w_T)\ \partial(\cdot)/\partial z$ and $D(\cdot)/Dt = \partial(\cdot)/\partial t + w\partial(\cdot)/\partial z$ are the material derivatives in a one-dimensional flow, following the hydrometeors and the moist air, respectively, and $r_w \approx \Sigma_k\, q_k$ is the condensed water mixing ratio.

The rate of variation of the water mixing ratio and the vapor mixing ratio are given by the conservation equations of mass (Bannon 2002)

$$\frac{Dr_w}{Dt_p} = \dot{r}_{v,\text{cond}}, \quad \text{and} \quad \frac{Dr_v}{Dt} = -\dot{r}_{v,\text{cond}}, \tag{11}$$

where $\dot{r}_{v,\text{cond}}$ is the rate of condensation. By substitution of (11) into (10), we obtain that the phase transitions term is given by $-w_T\dot{r}_{v,\text{cond}}$.

We estimate next the magnitude of $-w_T\dot{r}_{v,\text{cond}}$ in ideal conditions for representative clouds considering supersaturation, pressure, and temperature constants. For a $n(D)$ droplet population

$$\frac{Dr_w}{Dt_p} = \frac{1}{\rho_a} n \frac{Dm_p}{Dt_p} = 2\pi n \frac{\rho_w}{\rho_a} D \frac{S - 1}{F_k + F_d}, \tag{12}$$

where the rate of condensation is calculated with the simplified growth equation for one droplet (Mason 1971) of mass m_p, F_k represents the thermodynamic term that is associated with heat conduction, F_d the term associated with vapor diffusion, S is the ambient saturation ratio.

In the particular case of a water cloud, the terminal velocity of cloud droplets up to 80-μm diameter is given by the Stokes law, $w_T(\text{m s}^{-1}) = 1.19 \times 10^6(D(\text{cm})/2]^2$ (cf. Rogers and Yau 1989). Since the terminal velocity

TABLE 1. Phase acceleration term due to evaporation of hydrometeors in a subsaturated atmospheric layer with a saturation ratio S = 0.7. CET = central European time.

	Type of rainfall	N_0 ($\mathrm{m^{-3}\,mm^{-1}}$)	Λ ($\mathrm{mm^{-1}}$)	R ($\mathrm{mm\,h^{-1}}$)	$\lvert -\Sigma_k w_k (dr_k/dt_k) \rvert$ ($\times 10^{-5}\ \mathrm{m\,s^{-2}}$)
6 Jun 1968*	2205–2235 CET, thunderstorm*	35 000	3.7	10.2	6.5
	2235–2310 CET, thunderstorm*	4000	2.5	5.8	2.3
19 Jun 1969*	0510–0540 CET, shower*	16 000	3.8	4.0	2.8
	0550–0620 CET, widespread rain*	8000	2.6	8.0	4.2
Theoretical distributions	MP distribution	8000	$41.0R^{-0.21}$	10	4.5
	MP distribution	8000	$41.0R^{-0.21}$	50	12.0
	MP distribution	8000	$41.0R^{-0.21}$	100	17.0
	Hail**	52	0.33		3.1×10^{-3}

* Raindrop size distributions of MP type based on data of Waldvogel (1974).
** Typical hailstone size distribution based on data of Federer and Waldvogel (1975) (cf. Pruppacher and Klett 1980).

of cloud particles depends on the diameter square of the particle, in a cloud with a uniform size particle distribution the phase change term is proportional to the liquid water content of the cloud for given thermodynamic condition. For an ambient saturation ratio of S = 1.05, a temperature T = 273 K, a pressure of 80 kPa, and a liquid water content of 3 g $\mathrm{kg^{-1}}$, the order of magnitude of this term in typical clouds is $O(10^{-5})$ ($\mathrm{m\,s^{-2}}$).

2) EVAPORATION OF PRECIPITATION

To estimate the magnitude of the evaporation term we assume stationary precipitation and hydrometeor size distributions following Marshall–Palmer (MP) exponential spectra and crossing an atmospheric layer with relative humidity of 70% and $w = 0\ \mathrm{m\,s^{-1}}$. The hydrometeor velocity is calculated according to the following empirical formulas:

(i) rain case (Rogers and Yau 1989)

$$w_T\,(\mathrm{m\,s^{-1}}) = 8 \times 10^3[D(\mathrm{cm})/2], \quad 80\ \mu\mathrm{m} < D < 1.2\ \mathrm{mm}, \tag{13a}$$

$$w_T\,(\mathrm{m\,s^{-1}}) = 2.2 \times 10^3\sqrt{D(\mathrm{cm})/2}, \quad D > 1.2\ \mathrm{mm}; \tag{13b}$$

(ii) hail case (cf. Pruppacher and Klett 1980)

$$w_T\,(\mathrm{m\,s^{-1}}) = 9[D(\mathrm{cm})]^{0.8}, \quad 0.1\ \mathrm{cm} < D < 8\ \mathrm{cm}. \tag{13c}$$

The rate of evaporation is calculated by the growth equation (Mason 1971) and the phase change term is given by

$$-\sum_k w_k \frac{Dr_k}{Dt_k} = 2\pi \frac{\rho_P}{\rho_k} \frac{S-1}{F_k + F_d} \int_{D_{\min}}^{D_{\max}} D w_T(D) N(D)\, dD, \tag{14}$$

where D is the hydrometeor equivalent diameter and $N(D)dD = N_0 e^{-\Lambda D} dD$ is the number of drops per unit volume with diameters between D and $D + dD$. Table 1 gives the characteristics properties of the raindrop spectra simulated as well as the value of the phase change term for each case. The table shows that

$$-\sum_k w_k (Dr_k/Dt_k) < O(10^{-4})\ (\mathrm{m\,s^{-2}}). \tag{15}$$

Because the contribution of the phase change term, (15), is negligible with respect to the gravity term, (9), (7) shows that the commonly used expression for virtual temperature, (1), represents well the impact of the presence of water on the vertical air movement.

5. Conclusions

Herein clouds have been considered as multicomponent and multiphase systems whose dynamical state is represented by the equation of motion of the hydrometeors and of the saturated air (9). From this last equation, the equation of state of ideal gases, the hydrostatic equilibrium hypothesis, and the definition of virtual temperature for humid air, we obtained the temperature that dry air should have in order to experience the same vertical acceleration than the saturated air in clouds. We call it the dynamic virtual temperature of the parcel. It depends on the vapor mixing ratio, the liquid and/or solid water content, and phase transitions of the hydrometeors.

The analysis of the order of magnitude of the water terms shows that in a nonturbulent flow the phase transition terms are negligible. For this reason, the dynamic virtual temperature is formally identical to the expression of the cloudy virtual temperature. According to this last, a cloud is seen as a mixture of three gases: dry air, water vapor, and a hydrometeor-pseudo-gas, all three moving at the same velocity. Buoyancy is then associated with the density of the mixture. From the point of view of the dynamic virtual temperature, changes in buoyancy are due to the drag force between the hydrometeors and the moist air because of their relative motion. However, during their fall, hydrometeors quickly reach equilibrium and because the pressure gradient force is negligible, the drag force is practically equal and opposite to the gravity force. With respect to calculations, this condition is equivalent to a change of density of the air parcel equal to the partial density of the included hydrometeors. That is why (1) and (7) are numerically equivalent. It is rather the physical interpretation commonly given to the cloudy temperature that is at fault. It does not represent the temperature at which dry air has the same density as cloud air at the same pressure, but rather the temperature at which dry air experiences the same acceleration as cloud air.

Furthermore, it is important to notice that in a quasi-hydrostatic atmosphere the pressure gradient force acting on the air parcel does not depend on the loading of hydrometeors. The pressure gradient is function of the density of the surrounding humid air, namely of its virtual temperature. However, the errors due to the use of a virtual temperature accounting for the water load in the calculation of the pressure gradient as suggested in the literature are of the same order of magnitude as the errors due to the measurement of the temperature itself.

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